4 Signal Space Representation of Waveforms

Signal space (or vector) representation of signals (waveforms) is a very effective and useful tool in the analysis of digitally modulated signals. In fact, any set of signals is equivalent to a set of vectors.

4.1 Review of Vector Space Concepts

Definition 4.1. The **inner product** of two (potentially complex-valued) n-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^H \boldsymbol{u}$$

$\langle \vec{u}, \vec{v} \rangle = \vec{v}^H \vec{u} = \sum_k u_k v_k^*$

where $(\cdot)^H$ denotes the **Hermitian transpose** operator which performs transposing operation and then conjugation.

4.2. Some properties of the inner product

$\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$

- $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle^*$

  $z + z^* = 2\,\mathrm{Re}\{z\} = 2\,\mathrm{Re}\{z^*\}$
- $\langle \boldsymbol{z}, \boldsymbol{w} \rangle + \langle \boldsymbol{w}, \boldsymbol{z} \rangle = 2\,\mathrm{Re}\{\langle \boldsymbol{z}, \boldsymbol{w} \rangle\} = 2\,\mathrm{Re}\{\langle \boldsymbol{w}, \boldsymbol{z} \rangle\}$.

Definition 4.3. Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **orthogonal** if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$.

More generally, a set of m vectors $\boldsymbol{v}^{(k)}$, $1 \le k \le m$, are **orthogonal** if $\left\langle \boldsymbol{v}^{(i)}, \boldsymbol{v}^{(j)} \right\rangle = 0$ for all $1 \le i, j \le m$, and $i \ne j$.

Definition 4.4. The **norm** of a vector $\boldsymbol{v}$ is denoted by $\|\boldsymbol{v}\|$ and is defined as

$$\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$$

which in the n-dimensional space is simply the **length** of the vector.

Definition 4.5. A set of m vectors is said to be **orthonormal** if the vectors are orthogonal and each vector has a unit norm.

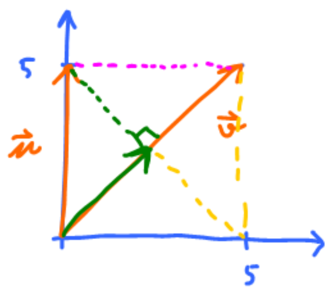
4.6. Given two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, we can decompose $\boldsymbol{v}$ into a sum of two vectors, one a multiple of $\boldsymbol{u}$ and the other orthogonal to $\boldsymbol{u}$.

(a) $\mathrm{proj}_{\boldsymbol{u}}(\boldsymbol{v}) = \frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle} \boldsymbol{u}$ is the orthogonal projection of $\boldsymbol{v}$ onto $\boldsymbol{u}$.

(b) $\boldsymbol{v} - \mathrm{proj}_{\boldsymbol{u}}(\boldsymbol{v})$ is the component of $\boldsymbol{v}$ orthogonal to $\boldsymbol{u}$.

$\vec{v} = \vec{v}_1 + \vec{v}_2$

Example 4.7. Let $\boldsymbol{v} = (5,5)^T$ and $\boldsymbol{u} = (0,5)^T$.



$$\text{proj}_{\vec{u}}\vec{v} = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u} = \frac{25}{25}\vec{u} = \vec{u} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v} = \frac{\cancel{25}}{\underset{2}{\cancel{50}}}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$$

4.8. A vector may also be represented as a linear combination of orthogonal unit vectors or an orthonormal basis $\{\boldsymbol{e}_{\cancel{i}}^{(i)}, 1 \le i \le n\}$, i.e.,

$$\boldsymbol{v} = \sum_{i=1}^{n} v_i \boldsymbol{e}_{\cancel{i}}^{(i)} \qquad \text{proj}_{\vec{e}}\vec{v} = \frac{\langle \vec{v}, \vec{e} \rangle}{\underset{\hookrightarrow 1}{\cancel{\langle \vec{e}, \vec{e} \rangle}}}\vec{e}$$

where, by definition, a unit vector has length unity and v_i is the projection of the vector $\boldsymbol{v}$ onto the unit vector $\boldsymbol{e}_i$, i.e.,

GSOP

$$v_i = \langle \boldsymbol{v}, \boldsymbol{e}_i \rangle$$

4.9. **Gram-Schmidt orthogonalization procedure** for constructing a set of orthonormal vectors from a set of n-dimensional vectors $\boldsymbol{v}^{(i)}$, $1 \le i \le M$.

(a) Arbitrarily select a vector from the set, say, $\boldsymbol{v}^{(1)}$. Let $\boldsymbol{u}^{(1)} = \boldsymbol{v}^{(1)}$. Normalize its length to obtain the first vector, say,

$$\boldsymbol{e}^{(1)} = \frac{\boldsymbol{u}^{(1)}}{\left\|\boldsymbol{u}^{(1)}\right\|}.$$

(b) Select an unselected vector from the set, say, $\boldsymbol{v}^{(2)}$. Subtract the projection of $\boldsymbol{v}^{(2)}$ onto $\boldsymbol{u}^{(1)}$:

$$\begin{aligned}\boldsymbol{u}^{(2)} &= \boldsymbol{v}^{(2)} - \text{proj}_{\boldsymbol{u}^{(1)}}\left(\boldsymbol{v}^{(2)}\right) = \boldsymbol{v}^{(2)} - \frac{\left\langle \boldsymbol{v}^{(2)}, \boldsymbol{u}^{(1)} \right\rangle}{\left\langle \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(1)} \right\rangle}\boldsymbol{u}^{(1)} \\ &= \boldsymbol{v}^{(2)} - \left\langle \boldsymbol{v}^{(2)}, \boldsymbol{e}^{(1)} \right\rangle \boldsymbol{e}^{(1)}.\end{aligned}$$

Then, we normalize the vector $\boldsymbol{u}^{(2)}$ to unit length:

$$\boldsymbol{e}^{(2)} = \frac{\boldsymbol{u}^{(2)}}{\left\|\boldsymbol{u}^{(2)}\right\|}.$$

(c) Continue by selecting an unselected vector from the set, say, $\boldsymbol{v}^{(3)}$ and subtract the projections of $\boldsymbol{v}^{(3)}$ into $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$:

$$
\begin{aligned}
\boldsymbol{u}^{(3)} &= \boldsymbol{v}^{(3)} - \operatorname{proj}_{\boldsymbol{u}^{(1)}}\left(\boldsymbol{v}^{(3)}\right) - \operatorname{proj}_{\boldsymbol{u}^{(2)}}\left(\boldsymbol{v}^{(3)}\right) \\
&= \boldsymbol{v}^{(3)} - \frac{\left\langle \boldsymbol{v}^{(3)}, \boldsymbol{u}^{(1)} \right\rangle}{\left\langle \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(1)} \right\rangle}\boldsymbol{u}^{(1)} - \frac{\left\langle \boldsymbol{v}^{(3)}, \boldsymbol{u}^{(2)} \right\rangle}{\left\langle \boldsymbol{u}^{(2)}, \boldsymbol{u}^{(2)} \right\rangle}\boldsymbol{u}^{(2)} \\
&= \boldsymbol{v}^{(3)} - \left\langle \boldsymbol{v}^{(3)}, \boldsymbol{e}^{(1)} \right\rangle \boldsymbol{e}^{(1)} - \left\langle \boldsymbol{v}^{(3)}, \boldsymbol{e}^{(2)} \right\rangle \boldsymbol{e}^{(2)}.
\end{aligned}
$$

Then, we normalize the vector $\boldsymbol{u}^{(3)}$ to unit length:

$$
\boldsymbol{e}^{(3)} = \frac{\boldsymbol{u}^{(3)}}{\left\|\boldsymbol{u}^{(3)}\right\|}.
$$

By continuing this procedure, we construct a set of N orthonormal vectors, where

$$
N \le \min(M, n).
$$

Example 4.10. Consider four vectors: $\boldsymbol{v}^{(1)} = (1,1,0)^T$, $\boldsymbol{v}^{(2)} = (1,-1,0)^T$, $\boldsymbol{v}^{(3)} = (1,1,-1)^T$ and $\boldsymbol{v}^{(4)} = (-1,-1,-1)^T$.

Simple solution:

$\vec{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\vec{e}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{v}^{(1)} = \vec{e}^{(1)} + \vec{e}^{(2)}$

$\vec{v}^{(2)} = \vec{e}^{(1)} - \vec{e}^{(2)}$

$\vec{v}^{(3)} = \vec{e}^{(1)} + \vec{e}^{(2)} - \vec{e}^{(3)}$

$\vec{v}^{(4)} = -\vec{e}^{(1)} - \vec{e}^{(2)} - \vec{e}^{(3)}$

$\vec{u}^{(1)} = \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{e}^{(1)} = \dfrac{\vec{u}^{(1)}}{\|\vec{u}^{(1)}\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{u}^{(2)} = \vec{v}^{(2)} - \underbrace{\operatorname{proj}_{\vec{u}^{(1)}} \vec{v}^{(2)}}_{\frac{\langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} = \vec{0}} = \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

0

$\vec{e}^{(2)} = \dfrac{\vec{u}^{(2)}}{\|\vec{u}^{(2)}\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\vec{u}^{(3)} = \vec{v}^{(3)} - \underbrace{\operatorname{proj}_{\vec{u}^{(1)}} \vec{v}^{(3)}}_{\frac{2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} - \underbrace{\operatorname{proj}_{\vec{u}^{(2)}} \vec{v}^{(3)}}_{\vec{0}} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$\vec{e}^{(3)} = \vec{u}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$\cancel{\vec{u}^{(4)}} = \vec{v}^{(4)} - \operatorname{proj}_{\vec{u}^{(1)}} \vec{v}^{(4)} - \operatorname{proj}_{\vec{u}^{(2)}} \vec{v}^{(4)} - \operatorname{proj}_{\vec{u}^{(3)}} \vec{v}^{(4)} = \vec{0}$

GSOP Start with $\vec{v}^{(1)}, \vec{v}^{(2)}, \ldots, \vec{v}^{(M)}$ in the ex. $M=4$ in the Quiz $M=6$

$$\vec{v}^{(1)}, \vec{v}^{(2)}, \ldots, \vec{v}^{(M)} \xrightarrow{\text{GSOP}} \vec{w}^{(1)}, \vec{w}^{(2)}, \ldots, \vec{w}^{(K)} \xrightarrow{\text{normalization}} \vec{e}^{(1)}, \vec{e}^{(2)}, \ldots, \vec{e}^{(K)}$$

① $\vec{v}^{(j)}$ can be expressed as $\sum_{i=1}^{K} c_i^{(j)} \vec{e}^{(i)}$

In fact $\vec{v}^{(j)} = \sum_{i=1}^{a_j} c_i^{(j)} \vec{e}^{(i)}$ for some $a_j \le j$

Ex. $\vec{v}^{(1)} = \begin{pmatrix}1\\1\\0\end{pmatrix} = 1\vec{w}^{(1)} + 0\vec{w}^{(2)} + 0\vec{w}^{(3)} = \sqrt{2}\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 0\,\vec{e}^{(3)} = \underbrace{\begin{bmatrix}\vec{e}^{(1)} & \vec{e}^{(2)} & \vec{e}^{(3)}\end{bmatrix}}_{E} \underbrace{\begin{pmatrix}\sqrt{2}\\0\\0\end{pmatrix}}_{\vec{c}^{(1)}}$

$\vec{v}^{(2)} = \begin{pmatrix}1\\-1\\0\end{pmatrix} = 0\vec{w}^{(1)} + 1\vec{w}^{(2)} + 0\vec{w}^{(3)} = 0\,\vec{e}^{(1)} + \sqrt{2}\,\vec{e}^{(2)} + 0\,\vec{e}^{(3)} = E\,\vec{c}^{(2)}$ $\hookrightarrow \begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix}$

$\vec{v}^{(3)} = \begin{pmatrix}1\\1\\-1\end{pmatrix} = 1\vec{w}^{(1)} + 0\vec{w}^{(2)} + 1\vec{w}^{(3)} = \sqrt{2}\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 1\,\vec{e}^{(3)} = E\,\vec{c}^{(3)}$

$\vec{v}^{(4)} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix} = -1\vec{w}^{(1)} + 0\vec{w}^{(2)} + 1\vec{w}^{(3)} = -\sqrt{2}\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 1\,\vec{e}^{(3)} = E\,\vec{c}^{(4)}$

$$C = \begin{bmatrix}\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2}\\ 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}\vec{v}^{(1)} & \vec{v}^{(2)} & \vec{v}^{(3)} & \vec{v}^{(4)}\end{bmatrix}}_{V} = \begin{bmatrix}E\vec{c}^{(1)} & E\vec{c}^{(2)} & E\vec{c}^{(3)} & E\vec{c}^{(4)}\end{bmatrix} = E\underbrace{\begin{bmatrix}\vec{c}^{(1)} & \vec{c}^{(2)} & \vec{c}^{(3)} & \vec{c}^{(4)}\end{bmatrix}}_{C}$$

$$V = EC$$

upper triangular

unitary

$EE^H = I$

$E^H E = I$

unitary (complex-valued matrix)

orthogonal (real-valued ..)

"Similar" function in MATLAB:

QR-decomposition

Given an $m \times n$ matrix A,

find $m \times m$ unitary matrix Q and

$m \times n$ upper triangular matrix R

such that $A = QR$

MATLAB command $[Q, R] = qr(A)$

② Geometric Conservation

Same inner product

$$\langle \vec{v}^{(i)}, \vec{v}^{(j)} \rangle = (\vec{v}^{(j)})^H \vec{v}^{(i)} = (E\vec{c}^{(j)})^H (E\vec{c}^{(i)})$$

$$= (\vec{c}^{(j)})^H \underbrace{E^H E}_{I} \vec{c}^{(i)} = \langle \vec{c}^{(i)}, \vec{c}^{(j)} \rangle$$

Same norm

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\langle \vec{c}, \vec{c} \rangle} = \|\vec{c}\|$$

Same distance

$$d(\vec{v}^{(i)}, \vec{v}^{(j)}) \equiv \|\vec{v}^{(j)} - \vec{v}^{(i)}\| = \|\vec{c}^{(j)} - \vec{c}^{(i)}\| = d(\vec{c}^{(j)}, \vec{c}^{(i)})$$

distance

$$d(\vec{v}^{(1)}, \vec{v}^{(2)}) = \|\vec{v}^{(2)} - \vec{v}^{(1)}\|$$

4.2 Signal Space Concepts

As in the case of vectors, we may develop a parallel treatment for a set of signals.

Definition 4.11. The **inner product** of two generally complex-valued signals $x_1(t)$ and $x_2(t)$ is denoted by $\langle x_1(t), x_2(t)\rangle$ and defined by

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt.$$

The signals are **orthogonal** if their inner product is zero.

The **norm** of a signal is defined as

$$\|x(t)\| = \sqrt{\langle x(t), x(t)\rangle} = \sqrt{\mathcal{E}_x}$$

where $\mathcal{E}_x$ is the energy in $x(t)$.

A set of m signal is **orthonormal** if they are orthogonal and their norms are all unity.

4.12. Similar to 4.9, the **Gram-Schmidt orthogonalization procedure** can be used to construct a set of orthonormal waveforms from a set of finite energy signal waveforms: $x_i(t)$, $1 \leq i \leq m$.

Once we have constructed the set of, say K, orthonormal waveforms $\{\phi_k(t)\}$, we can express the signals $\{x_i(t)\}$ as linear combinations of the $\phi_k(t)$. Thus, we may write

$$x_i(t) = \sum_k x_{i,k}\phi_k(t).$$

Based on the above expression, each signal may be represented by the vector (or sequence)

$$\boldsymbol{x}^{(i)} = (x_{i,1}, x_{i,2}, \ldots, x_{i,K})^T,$$

or, equivalently, as a point in the N-dimensional (in general, complex) signal space.

Definition 4.13. As discussed in 4.12, a set of M signals $\{x_i(t)\}$ can be represented by a set of M vectors $\left\{\boldsymbol{x}^{(i)}\right\}$ in the K-dimensional space. The corresponding set of vectors is called the **signal space representation**, or **constellation**, of $\{x_i(t)\}$.

- If the original signals are real, then the corresponding vector representations are in $\mathbb{R}^N$; and if the signals are complex, then the vector representations are in $\mathbb{C}^N$.
- Figure 2 demonstrates the process of obtaining the vector equivalent from a signal (signal-to-vector mapping) and vice versa (vector-to-signal mapping).

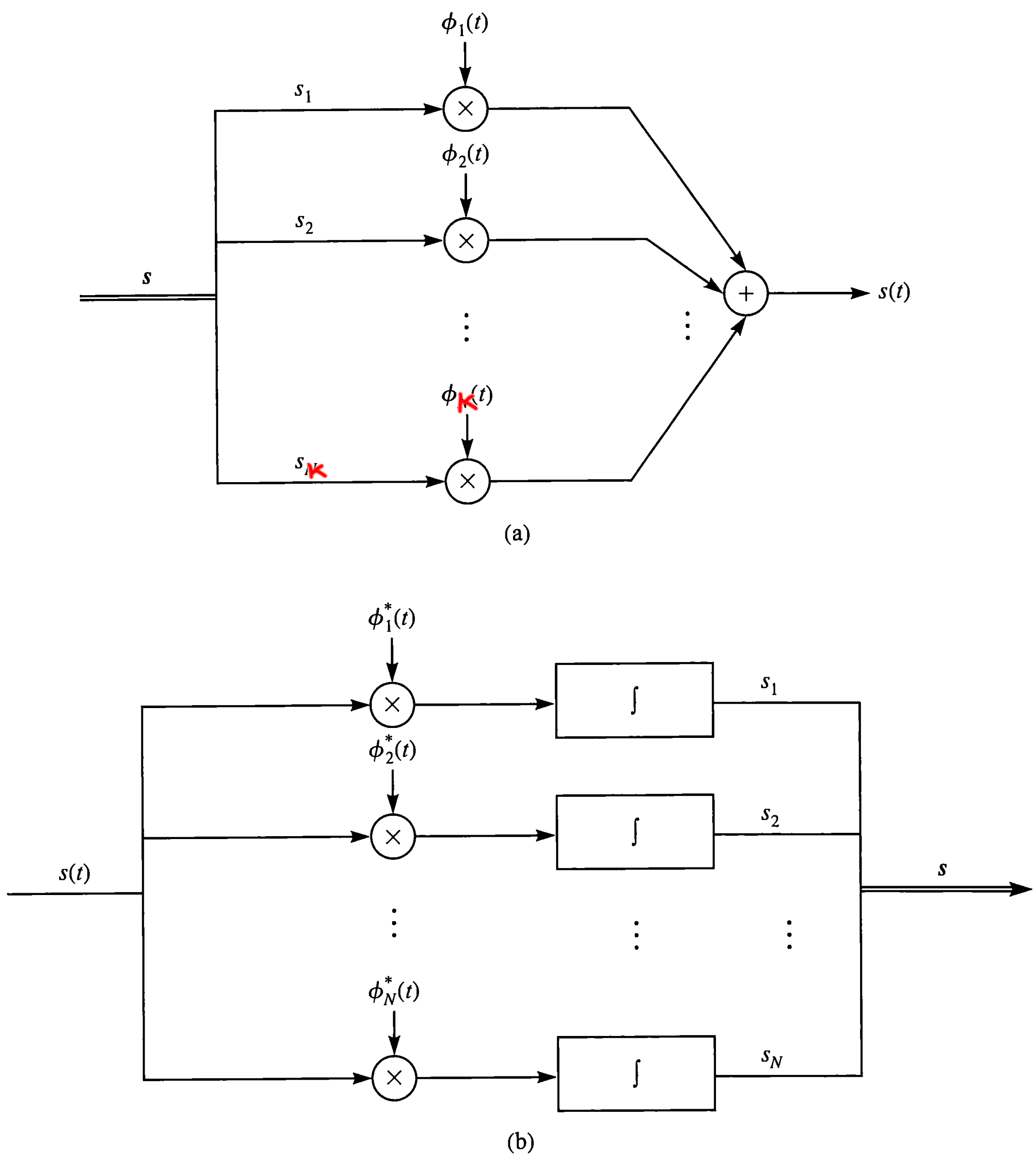


Figure 2: Vector to signal (a), and signal to vector (b) mappings.

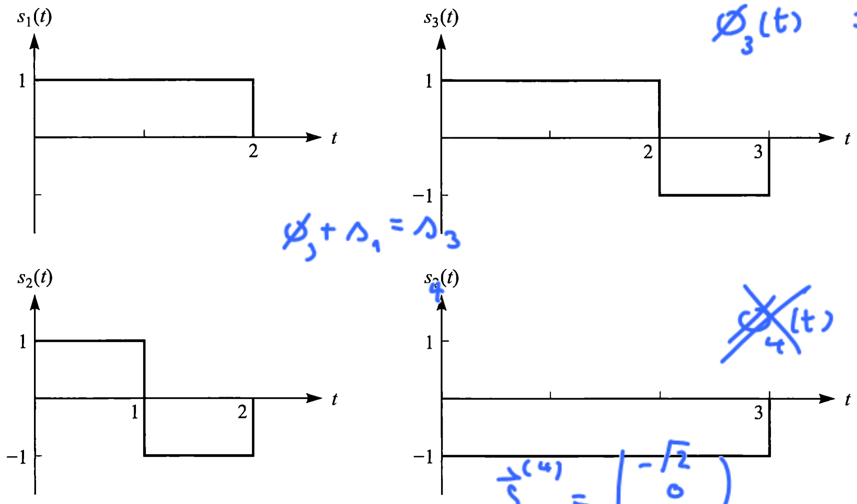
Example 4.14. Consider the four waveforms illustrated in Figure 3.

$\phi_1(t) = \frac{1}{\sqrt{2}} s_1(t)$

$\vec{s}^{(1)} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$

$\phi_2(t) = \frac{1}{\sqrt{2}} s_2(t)$

$\vec{s}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$



$\phi_3 + s_1 = s_3$

$\phi_3(t) =$

$\vec{s}^{(3)} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}$

$\cancel{\phi_4}(t) =$

$\vec{s}^{(4)} = \begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$

$= \vec{s}^{(1)} + \vec{\phi}^{(3)} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Figure 3: Four signals for orthogonalization in Example 4.14

4.15. From the orthonormality of the basis, we have

(a) the inner product of two signals is equal to the inner product of the corresponding vectors:

$$\langle x_i(t), x_j(t) \rangle = \left\langle \boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)} \right\rangle.$$

(b) $\mathcal{E}_{x^{(k)}} = \|x_k(t)\|^2 = \left\|\boldsymbol{x}^{(k)}\right\|^2.$

4.16. It should be emphasized, however, that the functions $\{\phi_k(t)\}$ obtained from the Gram-Schmidt procedure are not unique. If we alter the order in which the orthogonalization of the signals $\{x_i(t)\}$ is performed, the orthonormal waveforms will be different and the corresponding vector representation of the signals $\{x_i(t)\}$ will depend on the choice of the orthonormal functions $\{\phi_k(t)\}$. Nevertheless, the dimensionality of the signal space N will not change, and the vectors $\boldsymbol{x}^{(i)}$ will retain their geometric configuration; i.e., their lengths and their inner products will be invariant to the choice of the orthonormal functions $\{\phi_k(t)\}$.