4.4 Energy and Power

Definition 4.27. For a signal $g(t)$, the instantaneous power $p(t)$ dissipated in the 1-Ω resister is $p_g(t) = |g(t)|^2$ regardless of whether $g(t)$ represents a voltage or a current. To emphasize the fact that this power is based upon unity resistance, it is often referred to as the **normalized power**.

Definition 4.28. The total (normalized) ***energy*** of a signal $g(t)$ is given by

$$E_g = \int_{-\infty}^{+\infty} p_g(t)dt = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 dt.$$

4.29. By the **Parseval's theorem** discussed in 2.39, we have

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df.$$

Definition 4.30. The average (normalized) ***power*** of a signal $g(t)$ is given by

general formula always work but difficult to apply.

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 dt. = \langle |g(t)|^2 \rangle$$

Definition 4.31. To simplify the notation, there are two operators that used angle brackets to define two frequently-used integrals:

Some properties for the average operator

(a) The "**time-average**" operator:

① $g(t) \equiv 5 \quad \langle g \rangle = 5$

$\langle c \rangle = c$ for any constant c.

$$\langle g \rangle \equiv \langle g(t) \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)dt \qquad (46)$$

② $\langle a g_1 + b g_2 \rangle = a \langle g_1 \rangle + b \langle g_2 \rangle$

(b) The **inner-product** operator:

$$\langle g_1, g_2 \rangle \equiv \langle g_1(t), g_2(t) \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt \qquad (47)$$

Definition

When $\langle g_1, g_2 \rangle = 0$, we say g_1 and g_2 are orthogonal.

4.32. Using the above definition, we may write

- $E_g = \langle g, g \rangle = \langle G, G \rangle$ where $G = \mathcal{F}\{g\}$
- $P_g = \left\langle |g|^2 \right\rangle$

- Parseval's theorem: $\langle g_1, g_2\rangle = \langle G_1, G_2\rangle$
  where $G_1 = \mathcal{F}\{g_1\}$ and $G_2 = \mathcal{F}\{g_2\}$

4.33. Time-Averaging over Periodic Signal: For ***periodic*** signal $g(t)$ with period T_0, the time-average operation in (46) can be simplified to

$$\langle g\rangle = \frac{1}{T_0}\int_{T_0} g(t)dt$$

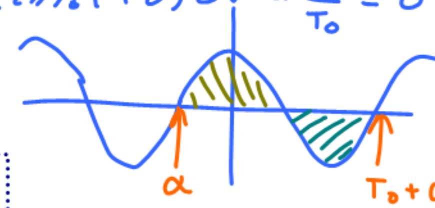
$\int_0^{T_0}$, $\int_1^{T_0+1}$, $\int_\alpha^{T_0+\alpha}$

where the integration is performed over a period of g.

Example 4.34. $\langle\cos(2\pi f_0 t+\theta)\rangle = \frac{1}{T_0}\int_{T_0}\cos(2\pi f_0 t+\theta)\,dt \overset{f_0\neq 0}{=} \frac{0}{T_0} = 0$

$= \begin{cases} 0, & f_0 \neq 0, \\ \cos\theta, & f_0 = 0. \end{cases}$

$f_0 = 0$: $= \cos\theta$

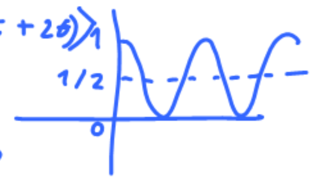


Similarly, $\langle\sin(2\pi f_0 t+\theta)\rangle = \begin{cases} 0, & f_0 \neq 0, \\ \sin\theta, & f_0 = 0. \end{cases}$

Example 4.35. $\left\langle\cos^2(2\pi f_0 t+\theta)\right\rangle = \left\langle \frac{1}{2}\left(1+\cos(2\pi 2f_0 t+2\theta)\right)\right\rangle$

$= \frac{1}{2} + \frac{1}{2}\begin{cases} 0, & f_0 \neq 0 \\ \cos 2\theta, & f_0 = 0 \end{cases}$

$= \begin{cases} 1/2, & f_0 \neq 0 \\ \cos^2\theta, & f_0 = 0 \end{cases}$



Example 4.36. $\left\langle e^{j(2\pi f_0 t+\theta)}\right\rangle = \langle\cos(2\pi f_0 t+\theta) + j\sin(2\pi f_0 t+\theta)\rangle$

$= \begin{cases} 0, & f_0 \neq 0 \\ e^{j\theta}, & f_0 = 0 \end{cases}$

In particular, $\langle e^{j\beta t}\rangle = \begin{cases} 0, & \beta \neq 0 \\ 1, & \beta = 0 \end{cases}$

Example 4.37. Suppose $g(t) = ce^{j2\pi f_0 t}$ for some (possibly complex-valued) constant c and (real-valued) frequency f_0. Find P_g.

Ex. $g(t) = je^{j10\pi t}$

$P_g = \langle |g(t)|^2\rangle = \langle |ce^{j2\pi f_0 t}|^2\rangle = \langle |c|^2\rangle = |c|^2$

$P_g = |j|^2 = 1$

4.38. When the signal $g(t)$ can be expressed in the form $g(t) = \sum_k c_k e^{j2\pi f_k t}$ and the f_k are distinct, then its (average) power can be calculated from

$$P_g = \sum_k |c_k|^2$$

"power of sum" = "sum of power"
of orthogonal signals

This is guaranteed when the signals do not overlap in the freq. domain.

$f_1 = 3$ $\quad$ $f_2 = 4$ $\quad$ $f_1 \neq f_2$

Example 4.39. Suppose $g(t) = \overset{c_1}{2}e^{j6\pi t} + \overset{c_2}{3}e^{j8\pi t}$. Find P_g.

$\Downarrow$

$P_g = |c_1|^2 + |c_2|^2$

$P_g = 2^2 + 3^2 = 4 + 9 = 13$

$f_1 = 3$ $\quad$ $f_2 = 3$

Example 4.40. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j6\pi t}$. Find P_g.

$f_1 = f_2$

$\Rightarrow$ terms can be combined

$= 5e^{j6\pi t}$

$P_g = 5^2 = 25$

Example 4.41. Suppose $g(t) = \cos\overbrace{(2\pi f_0 t + \theta)}^{B}$. Find P_g.

Here, there are several ways to calculate P_g. We can simply use Example 4.35. Alternatively, we can first decompose the cosine into complex exponential functions using the Euler's formula:

Assume $f_0 \neq 0$.

$P_g = |c_1|^2 + |c_2|^2 = \frac{1}{4}\times 1 + \frac{1}{4}\times 1 = \frac{2}{4} = \frac{1}{2}$

① $\left\langle |\cos(2\pi f_0 t + \theta)|^2 \right\rangle \underset{4.35}{=} \frac{1}{2}$

② $g(t) = \frac{1}{2}e^{j(2\pi f_0 t + \theta)} + \frac{1}{2}e^{-j(2\pi f_0 t + \theta)} = \underbrace{\frac{1}{2}e^{j\theta}}_{c_1} e^{j2\pi \overset{f_1}{f_0} t} + \underbrace{\frac{1}{2}e^{-j\theta}}_{c_2} e^{j2\pi \overset{f_2}{(-f_0)} t}$

4.42. The (average) power of a sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is

$$P_g = \begin{cases} \frac{1}{2}|A|^2, & f_0 \neq 0, \\ |A|^2 \cos^2\theta, & f_0 = 0. \end{cases}$$

This property means any sinusoid with nonzero frequency can be written in the form

$$g(t) = \sqrt{2P_g}\cos(2\pi f_0 t + \theta).$$

4.43. Extension of 4.42: Consider sinusoids $A_k \cos(2\pi f_k t + \theta_k)$ whose frequencies are positive and distinct. The (average) power of their sum

$$g(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$$

is

$$P_g = \frac{1}{2}\sum_k |A_k|^2.$$

Example 4.44. Suppose $g(t) = 2\cos\left(2\pi\sqrt{3}t\right) + 4\cos\left(2\pi\sqrt{5}t\right)$. Find P_g.

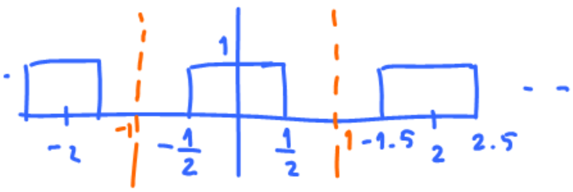
$\underset{\downarrow}{f_1}$ $\underset{\downarrow}{f_2}$

$P_g = \frac{2^2}{2} + \frac{4^2}{2} = \frac{1}{2}(4+16) = 10$

4.45. For ***periodic*** signal $g(t)$ with period T_0, there is also no need to carry out the limiting operation to find its (average) power P_g. We only need to find an average carried out over a single period:

$$P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2 dt.$$

$P_g = \frac{1}{2}\int_{-1}^{1} |g(t)|^2 dt = \frac{1}{2}\int_{-1/2}^{1/2} 1^2 dt = \frac{1}{2}$



(a) When the corresponding Fourier series expansion $g(t) = \sum\limits_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$ is known,

$$P_g = \sum_{k=-\infty}^{\infty} |c_k|^2$$

(b) When the signal $g(t)$ is real-valued and its (compact) trigonometric Fourier series expansion $g(t) = c_0 + 2\sum\limits_{k=1}^{\infty} |c_k| \cos\left(k\omega_0 t + \angle c_k\right)$ is known,

$$P_g = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2$$

Definition 4.46. Based on Definitions 4.28 and 4.30, we can define three distinct classes of signals:

(a) If E_g is finite and nonzero, g is referred to as an ***energy signal***.

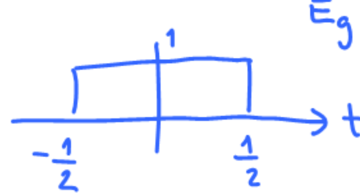
(b) If P_g is finite and nonzero, g is referred to as a ***power signal***.

(c) Some signals[17] are neither energy nor power signals.

- Note that the power signal has infinite energy and an energy signal has zero average power; thus the two categories are mutually exclusive.

Example 4.47. Rectangular pulse

$g(t) = \mathrm{rect}(t)$



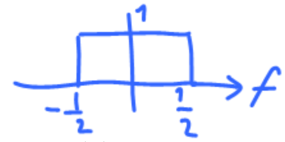
$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-1/2}^{1/2} 1^2 dt = 1$

is an energy signal

[17] Consider $g(t) = t^{-1/4} 1_{[t_0,\infty)}(t)$, with $t_0 > 0$.

$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df = 1$

Example 4.48. Sinc pulse



Example 4.49. For $\alpha > 0$, $g(t) = Ae^{-\alpha t} 1_{[0,\infty)}(t)$ is an energy signal with $E_g = |A|^2/2\alpha$.

Example 4.50. The rotating phasor signal $g(t) = Ae^{j(2\pi f_0 t + \theta)}$ is a power signal with $P_g = |A|^2$.

Example 4.51. The sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is a power signal with $P_g = |A|^2/2$.

4.52. Consider the transmitted signal

$$x(t) = m(t)\cos(2\pi f_c t + \theta)$$

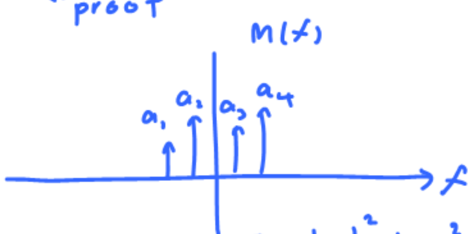
in DSB-SC modulation. Suppose $M(f - f_c)$ and $M(f + f_c)$ do not overlap (in the frequency domain).

(a) If $m(t)$ is a power signal with power P_m, then the average transmitted power is

$P_x = \left|\frac{a_1}{2}\right|^2 + \left|\frac{a_2}{2}\right|^2 + \cdots + \left|\frac{a_4}{2}\right|^2 + \left|\frac{a_1}{2}\right|^2 + \left|\frac{a_2}{2}\right|^2 + \cdots \left|\frac{a_4}{2}\right|^2$

$$P_x = \frac{1}{2}P_m. \quad = \frac{|a_1|^2}{2} + \frac{|a_2|^2}{2} + \cdots + \frac{|a_4|^2}{2}$$

"proof"



$P_m = |a_1|^2 + |a_2|^2 + \cdots + |a_4|^2$

(b) If $m(t)$ is an energy signal with energy E_m, then the transmitted energy is

$$E_x = \frac{1}{2}E_m.$$

- Q: Why is the power (or energy) reduced?

$m^2(t)\cos^2(\ )$

- Remark: When $x(t) = \sqrt{2}m(t)\cos(2\pi f_c t + \theta)$ (with no overlapping between $M(f - f_c)$ and $M(f + f_c)$), we have $P_x = P_m$.