CHAPTER 4

Circuit Theorems

The growth in areas of application of electrical circuits has led to an evolution from simple to complex circuits. To handle such complexity, engineers over the years have developed theorems to simplify circuit analysis. These theorems (Thevenin's and Norton's theorems) are applicable to *linear* circuits which are composed of resistors, voltage and current sources.

4.1. Linearity Property

Linear system

Input $x \longrightarrow \boxed{S} \longrightarrow$ Output $S(x)$

Ex $x \longrightarrow \boxed{} \longrightarrow x^2+1$

$f(x) = x^2+1$

DEFINITION 4.1.1. A **linear circuit** is a circuit whose output is linearly related (or directly proportional) to its input[1]. The input and output can be any voltage or current in the circuit. When we says that the input and output are linearly related, we mean they need to satisfies two properties:

(a) Homogeneous (Scaling): If the input is multiplied by a constant k, then we should observed that the output is also multiplied by k.

$$S(kx) = k\,S(x)$$

(b) Additive: If the inputs are summed then the output are summed.

$$S(x_1 + x_2) = S(x_1) + S(x_2)$$

[1]The input and output are sometimes referred to as cause and effect, respectively.

Q : Is function $f(x) = x^2 + 1$ linear?

A : No. ① $f(kx) = (kx)^2 + 1 = k^2x^2 + 1$

$\ne$ ← fail

$k f(x) = kx^2 + k$

Q : Is function $f(x) = 3x + 1$ linear?

A : No. $f(1) = 4$ $\quad f(2\times1) \ne 2f(1)$

$f(2) = 7$ $\quad f(1+1) \ne f(1) + f(1)$

It is actually called "affine" function.

1-D linear function : $\underbrace{f(x)}_{y} = ax$ one input, one output

single input, single output (SISO)

multi-dimensional

M-D linear function

value(s) of the source(s)

current/voltage value(s) that you are interested in

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = A \times \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

matrix

value of source ※1

value of source ※2

multiple inputs, multiple outputs (MIMO)

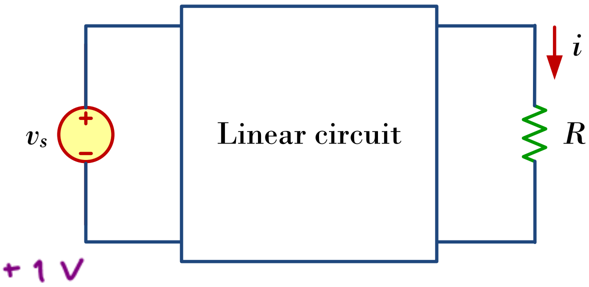
$$\underset{1\times1}{y} = (a_1 \ a_2 \cdots a_n) \times \underset{n\times1}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

multipul inputs, single output

EXAMPLE 4.1.2. The linear circuit below is excited by a voltage source v_s, which serves as the input.

Input : v_s Output i

$10\text{ V} \xrightarrow{\times \frac{1}{10}} 1\text{V}$



$2\text{ A} \xrightarrow{\times \frac{1}{10}} 0.2\text{ A}$

$11\text{ V} = 10\text{V} + 1\text{ V}$

$2 + 0.2 = 2.2\text{ A}$

The circuit is terminated by a **load** R. We take the current i through R as the output. Suppose $v_s = 10$ V gives $i = 2$ A. According to the linearity principle, $v_s = 1V$ will give $i = 0.2$ A. By the same token, $i = 1$ mA must be due to $v_s = 5$ mV.

$y = ax$

Of course, by Ohm's law, $v = Ri$ $i = \frac{1}{R} v$

EXAMPLE 4.1.3. A resistor is a **linear element** when we consider the current i as its input and the voltage v as its output because it has the following properties:

- Homogeneous (Scaling): If i is multiplied by a constant k, then the output v is multiplied by k.

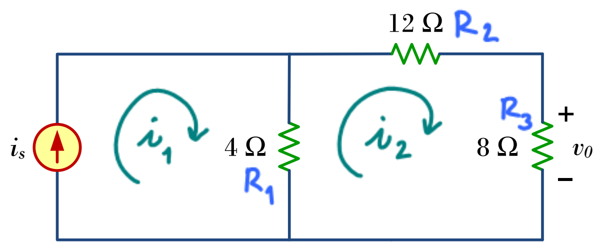
$$iR = v \Rightarrow (ki)R = kv$$

- Additive: If the inputs, i_1 and i_2, are summed then the corresponding output are summed.

$$i_1R = v_1, \ i_2R = v_2 \Rightarrow (i_1 + i_2)R = v_1 + v_2$$

4.1.4. Because $p = i^2R = v^2/R$ (making it a quadratic function rather than a linear one), the relationship between power and voltage (or current) is nonlinear. Therefore, the theorems covered in this chapter are not applicable to power.

EXAMPLE 4.1.5. For the circuit below, find v_o when (a) $i_s = 15$ and (b) $i_s = 30$.



Mesh 1

$i_1 = i_s$

Mesh 2

$-(i_2 - i_1)R_1 - i_2 R_2 - i_2 R_3 = 0$, with $i_1 = i_s$

$-R_1 i_2 + R_1 i_s - i_2 R_2 - i_2 R_3 = 0$

$i_2 = \dfrac{R_1 i_s}{R_1 + R_2 + R_3}$

$v_o = i_2 \times R_3 = \dfrac{R_1 R_3}{R_1 + R_2 + R_3} i_s$

$i_s = 15\,A$, $R_1 = 4\,\Omega$, $R_2 = 12\,\Omega$, $R_3 = 8\,\Omega$

$v_o = 20\,V$

(a) When $i_s = 15\,A$, $v_o = 20\,V$

$\times 2$ $\quad$ $\times 2$

(b) When $i_s = 30\,A$, $v_o = 40\,V$

Caution: The linear relationship is for input: i_s

output: v_o

not for input: R_3

output: v_o

$i_s = 15\,A$

$R_3 = 8\,\Omega \longrightarrow v_o = 20\,V$

$\times 2$

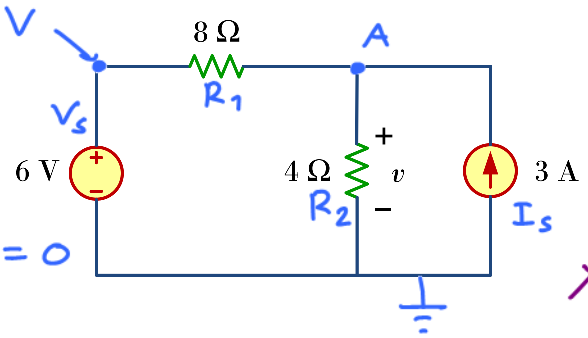
$R_3 = 16\,\Omega \longrightarrow v_o = 30\,V \neq 2 \times 20\,V$

4.2. Superposition

EXAMPLE 4.2.1. Find the voltage v in the following circuit.

Nodal analysis

KCL @ node A



$$\frac{V_A-6}{8}+\frac{V_A}{4}-3=0$$

$$\frac{V_A-V_s}{R_1}+\frac{V_A}{R_2}-I_s=0 \Rightarrow \underbrace{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}_{\frac{R_1+R_2}{R_1R_2}} V_A = \frac{V_s}{R_1}+I_s$$

$$V_A = \frac{R_1R_2}{R_1+R_2} I_s + \frac{R_2}{R_1+R_2} V_s$$

$$\underbrace{v}_{y} = V_A - 0 = V_A = \underbrace{\frac{R_1R_2}{R_1+R_2}}_{a_1}\underbrace{I_s}_{x_1} + \underbrace{\frac{R_2}{R_1+R_2}}_{a_2}\underbrace{V_s}_{x_2}$$

From the expression of v, observe that there are two contributions.

(a) When I_s acts alone (set $V_s = 0$), $v = \left(\frac{R_1R_2}{R_1+R_2}\right) I_s$

(b) When V_s acts alone (set $I_s = 0$), $v = \frac{R_2}{R_1+R_2} V_s$

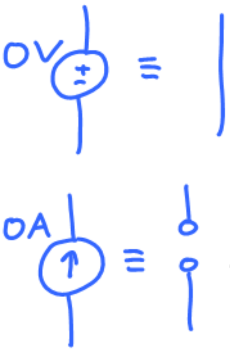
Key Idea: Find these contributions from the individual sources and then add them up to get the final answer.

DEFINITION 4.2.2. **Superposition** technique is a way to determine currents and voltages in a circuit that has multiple independent sources by considering the contribution of one source at a time and then add them up.

4.2.3. The **superposition principle** states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

4.2.4. To apply the superposition principle, we must keep two things in mind.

(a) We consider one independent source at a time while all other independent source are ***turned off***.[2]



- Replace other independent voltage sources by 0 V (or short circuits)
- Replace other independent current sources by 0 A (or open circuits)

This way we obtain a simpler and more manageable circuit.

(b) Dependent sources are left intact because they are controlled by circuit variable.

4.2.5. **Steps to Apply Superposition Principles:**

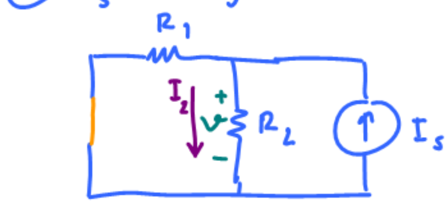
S1: Turn off all independent sources except one source.
Find the output due to that active source. (Here, you may use any technique of your choice.)

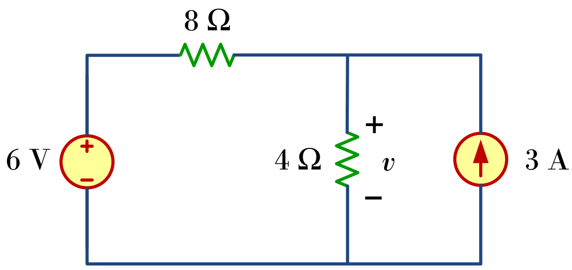
S2: Repeat S1 for each of the other independent sources.

S3: Find the total contribution by adding algebraically all the contributions due to the independent sources.

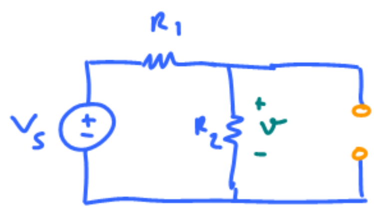
EXAMPLE 4.2.6. Back to Example 4.2.1.

① I_s acting alone





② V_s acting alone



$$I_2 = \frac{R_1}{R_1+R_2} I_s$$

current divider

$$v = I_2 \times R_2 = \frac{R_1 R_2}{R_1+R_2} I_s$$

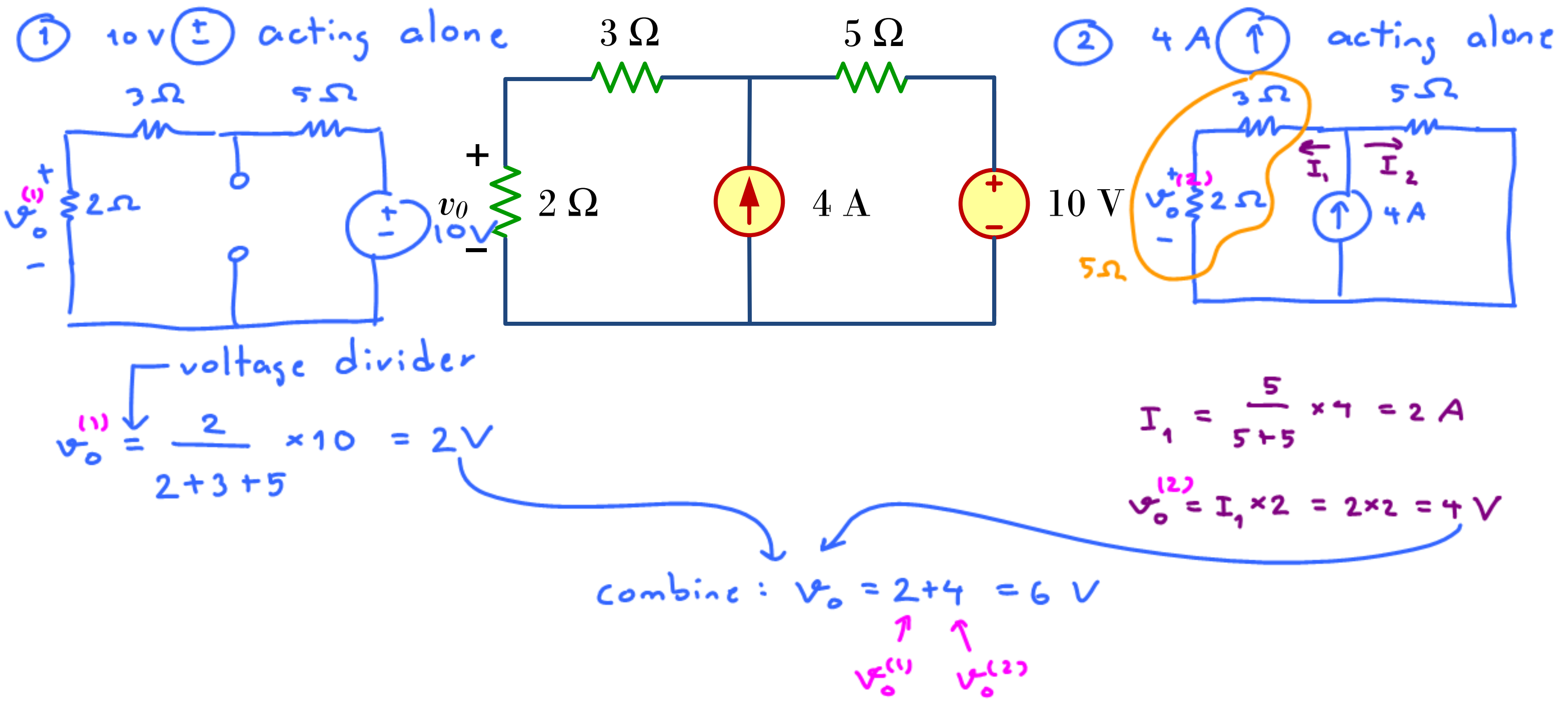
$$v = \frac{R_2}{R_1+R_2} V_s$$

voltage divider

$$v = \frac{R_1 R_2}{R_1+R_2} I_s + \frac{R_2}{R_1+R_2} V_s$$

[2]Other terms such as killed, made inactive, deadened, or set equal to zero are often used to convey the same idea.

Example 4.2.7. Using superposition theorem, find v_o in the following circuit.



4.2.8. Remark on linearity: Keep in mind that superposition is based on linearity. Hence, we cannot find the total power from the power due to each source, because the power absorbed by a resistor depends on the square of the voltage or current and hence it is not linear (e.g. because $5^2 \neq 1^2 + 4^2$).

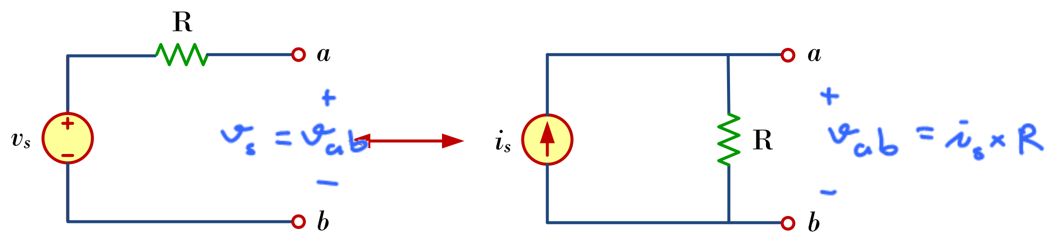
4.2.9. Remark on complexity: Superposition helps reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits.

However, it may very likely involve more work. For example, if the circuit has three independent sources, we may have to analyze three circuits. The advantage is that each of the three circuits is considerably easier to analyze than the original one.

4.3. Source Transformation

We have noticed that series-parallel resistance combination helps simplify circuits. The simplification is done by replacing one part of a circuit by its equivalence.[3] Source transformation is another tool for simplifying circuits.

4.3.1. A **source transformation** is the process of replacing a voltage source in series with a resistor R by a current source in parallel with a resistor R or vice versa.



Notice that when terminals $a - b$ are short-circuited, the short-circuit current flowing from a to b is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the righthand side. Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}. \tag{4.1}$$

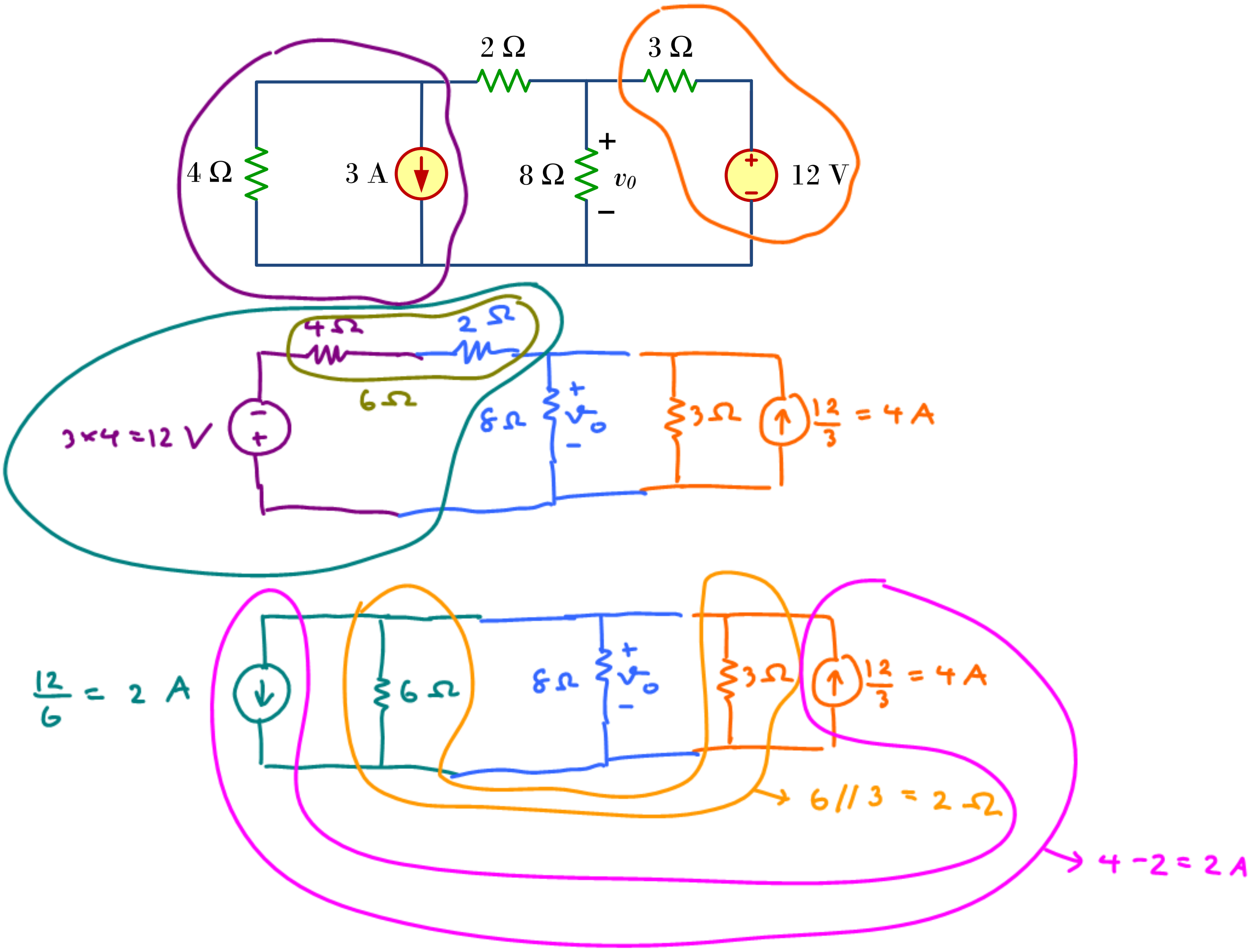
4.3.2. Remarks:

(a) The "R" in series with the voltage source and the "R" in parallel with the current source are not the same "resistor" even though they have the same value. In particular, the voltage values across them are generally different and the current values through them are generally different.

(b) From (4.1), an ideal voltage source with $R = 0$ cannot be replaced by a finite current source. Similarly, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source.

[3] Recall that an equivalent circuit is one whose $v - i$ characteristics are identical with the original circuit.

EXAMPLE 4.3.3. Use source transformation to find v_0 in the following circuit:

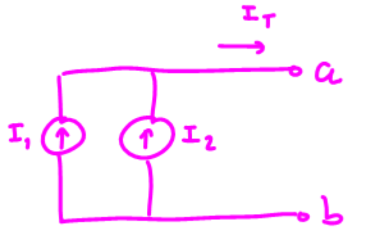


$$I_o = \frac{2}{2+8} \times 2 = \frac{2}{5}$$

$$v_o = I_o \times 8 = \frac{2}{5} \times 8 = \frac{16}{5}$$

$$= 3.2 \text{ V}$$

A note on combination of current sources

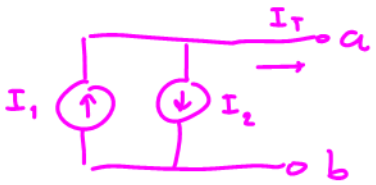


By KCL,

$$I_T - I_1 - I_2 = 0$$

$$I_T = I_1 + I_2$$

$\equiv$

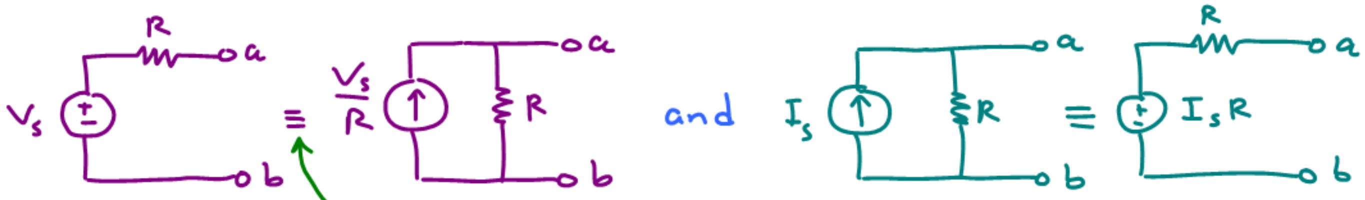


By KCL,

$$I_T - I_1 + I_2 = 0$$

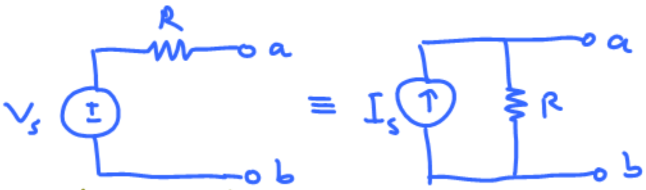
$$I_T = I_1 - I_2$$

Review: Source transformation

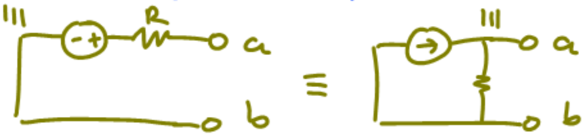


equivalent from the perspective of an external circuit that connects to the two terminals a-b

We summarize these as



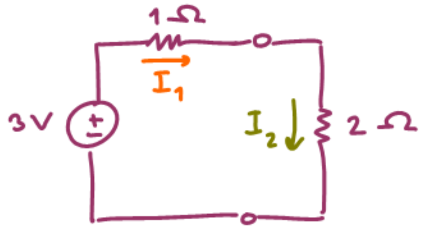
where $V_s = I_s R$.



Cautions:

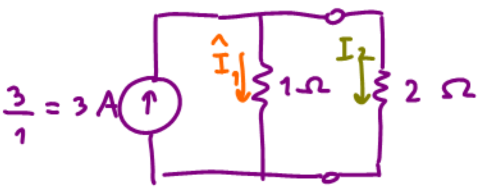
① The two resistors are not the same resistor even though they have the same resistance R.

Ex.



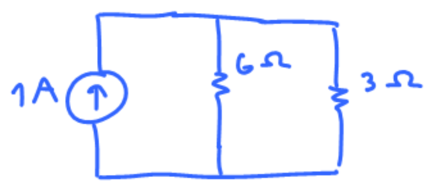
$I_1 = 1\text{ A}$ $\qquad$ $I_2 = 1\text{ A}$

current divider

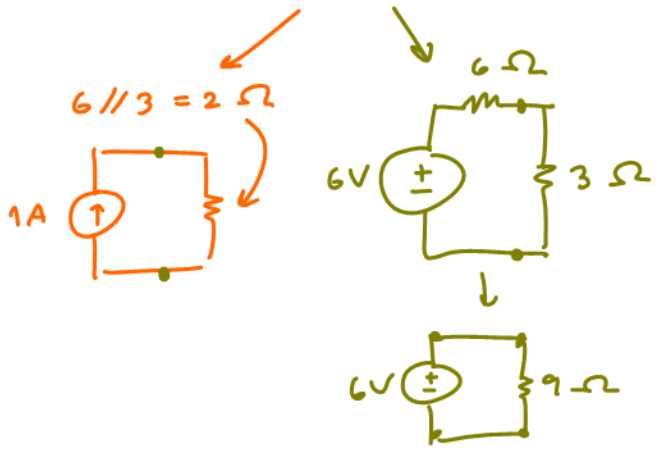


$\hat{I}_1 = \frac{2}{1+2} \times 3 = 2\text{ A}$ $\qquad$ $I_2 = \frac{1}{1+2} \times 3 = 1\text{ A}$

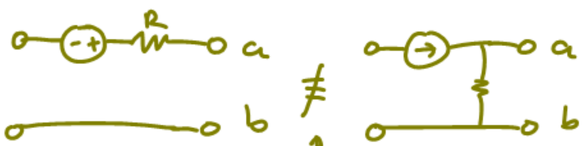
② Consider this circuit



$6 // 3 = 2\,\Omega$

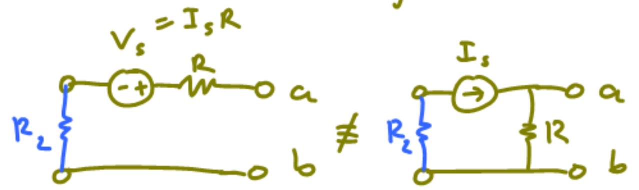


③



$\not\equiv$

in general



4.4. Thevenin's Theorem

4.4.1. It often occurs in practice that a particular element in a circuit or a particular part of a circuit is variable (usually called the **load**) while other elements are fixed.

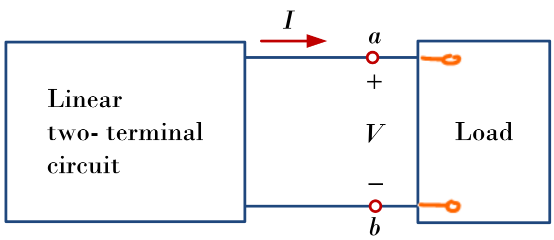
- As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load.

Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenins theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

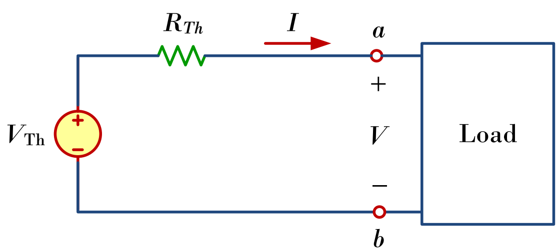
4.4.2. Thevenin's Theorem is an important method to simplify a complicated circuit to a very simple circuit. It states that a circuit can be replaced by an equivalent circuit consisting of an independent voltage source V_{Th} in series with a resistor R_{Th}, where

V_{Th}: the open circuit voltage at the terminal.

R_{Th}: the equivalent resistance at the terminals when the independent sources are turned off.



(a)



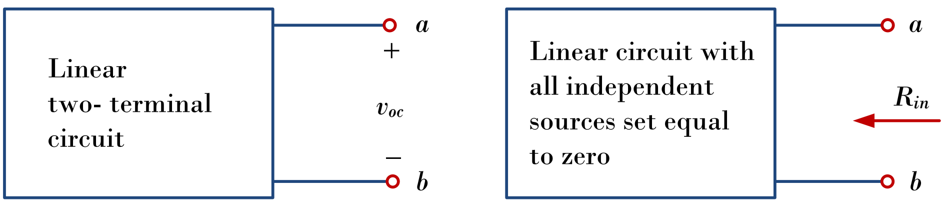
(b)

This theorem allows us to convert a complicated network into a simple circuit consisting of a single voltage source and a single resistor connected in series. The circuit is equivalent in the sense that it looks the same from the outside, that is, it behaves the same electrically as seen by an outside observer connected to terminals a and b.

4.4.3. **Steps to Apply Thevenin's theorem. (Case I: No dependent source)**

S1: Find R_{Th}: Turn off all independent sources. R_{Th} is the input resistance of the network looking between terminals a and b.

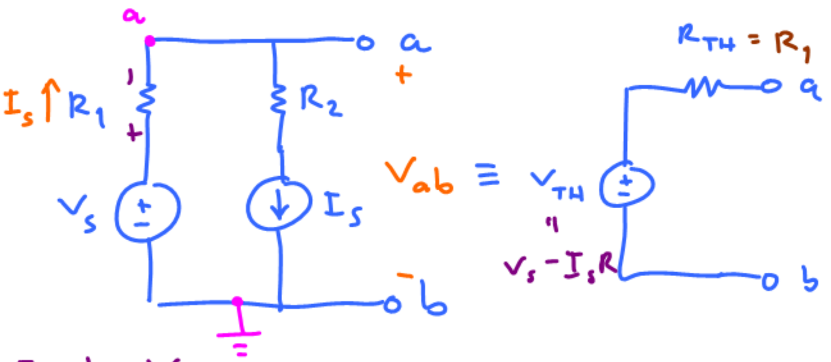
S2: Find V_{Th}: Open the two terminals (remove the load) which you want to find the Thevenin equivalent circuit. V_{Th} is the open-circuit voltage across the terminals.



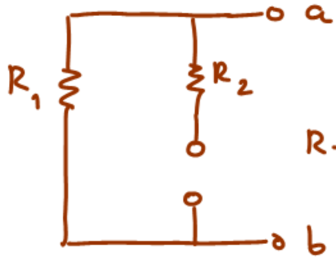
$V_{Th} = v_{oc}$ $R_{Th} = R_{in}$

S3: Connect V_{Th} and R_{Th} in series to produce the Thevenin equivalent circuit for the original circuit.

EXAMPLE 4.4.4. Find the Thevenin equivalent circuit of the circuit shown below, to the left of the terminals a-b.



Find R_{TH}



$R_{TH} = R_{ab}$
$= R_1$

Find V_{ab}

$b \to a$

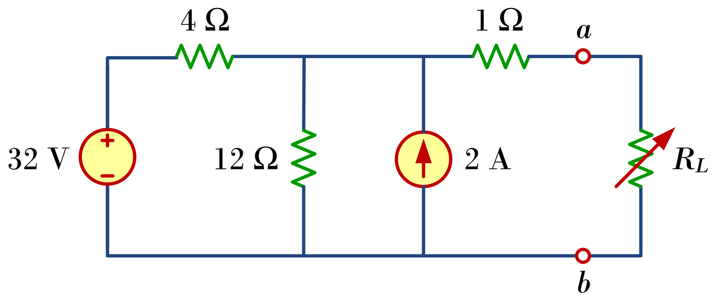
gain V_s

drop $I_s R_1$

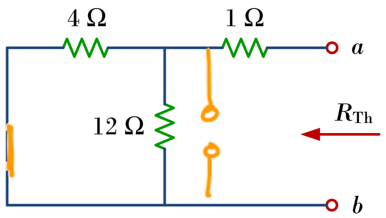
$V_{ab} = V_s - I_s R_1$
$= V_{TH}$

$$\frac{V_a - V_s}{R_1} + I_s = 0$$

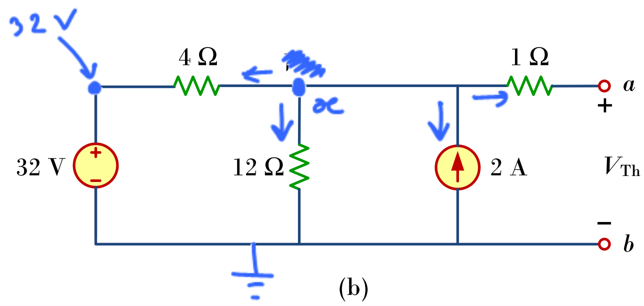
Example 4.4.5. Find the Thevenin equivalent circuit of the circuit shown below, to the left of the terminals a-b. Then find the current through $R_L = 6$, 16, and 36 Ω.



Solution:



(a)



(b)

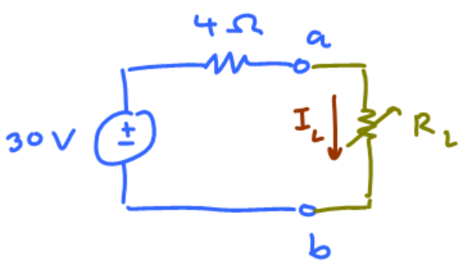
$$R_{TH} = (4 \,//\, 12) + 1$$
$$= 3 + 1 = 4\ \Omega$$

Nodal analysis

KCL @ 'a' $\quad \dfrac{V_a - V_x}{1} + 0 = 0 \Rightarrow V_a = V_x$

KCL @ 'x' $\quad \dfrac{V_x - 32}{4} + \dfrac{V_x}{12} - 2 + \cancel{\dfrac{V_x - V_a}{1}} = 0$

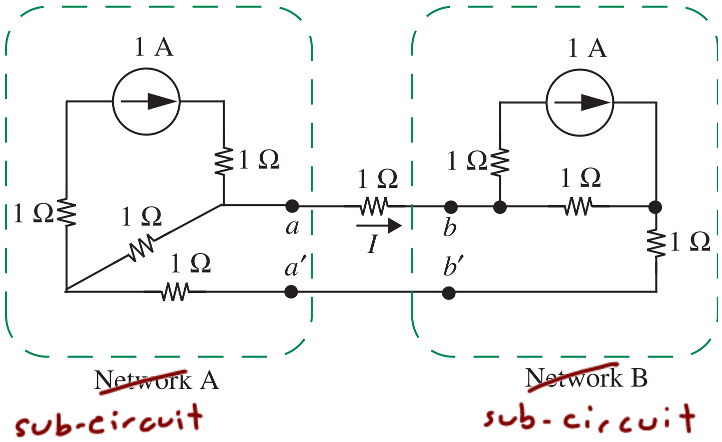
$V_x = 30\ V = V_{TH}$



$$I_L = \frac{30}{4 + R_L} \Rightarrow$$

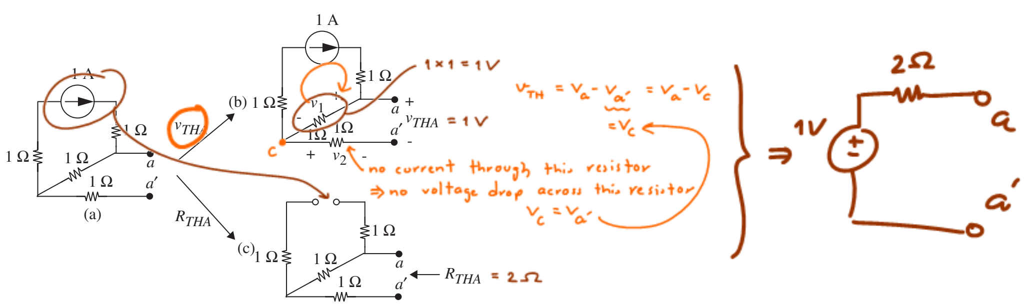
R_L	I_L
6	3 A
16	1.5 A
36	0.75 A

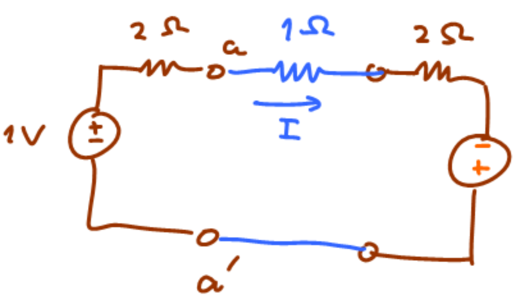
Example 4.4.6. Determine the current I in the branch ab in the circuit below.



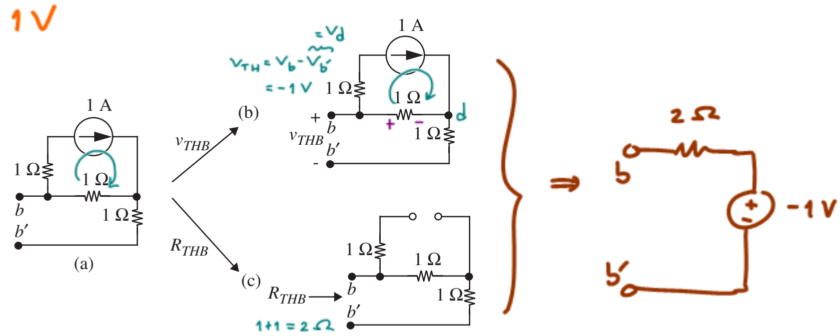
Solution:

There are many approaches that we can take to obtain the current I. For example, we could apply the node method and determine the node voltages at nodes a and b and thereby determine the current I. However, we will find the Thvenin equivalent network for the subcircuit to the left of the aa' terminal pair (Network A) and for the subcircuit to the right of the bb' terminal pair (Network B), and then using these subcircuits solve for the current I.





$$I = \frac{1+1}{2+1+2} = \frac{2}{5}\ A$$



4.4.7. **Steps to Apply Thevenin's theorem.(Case II: with dependent sources)**

S1: Find R_{TH}:

S1.1 Turn off all independent sources (but leave the dependent sources on).

S1.2.i Apply a voltage source v_o at terminals a and b, determine the resulting current i_o, then

$$R_{TH} = \frac{v_o}{i_o}$$

Note that: We usually set $v_o = 1$ V.

Or, equivalently,

S1.2.ii Apply a current source i_o at terminal a and b, find v_o, then $R_{TH} = \frac{v_o}{i_o}$

S2: Find V_{TH}, as the open-circuit voltage across the terminals.

S3: Connect R_{TH} and V_{TH} in series.

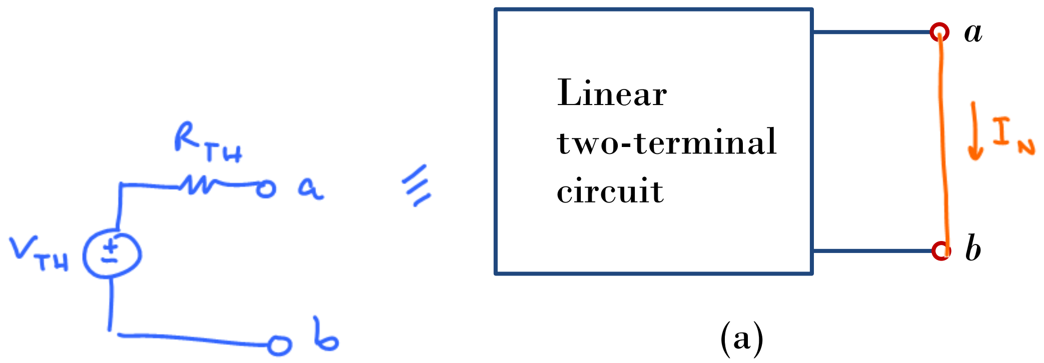
Remark: It often occurs that R_{TH} takes a negative value. In this case, the negative resistance implies that the circuit is supplying power. This is possible in a circuit with dependent sources.

4.5. Norton's Theorem

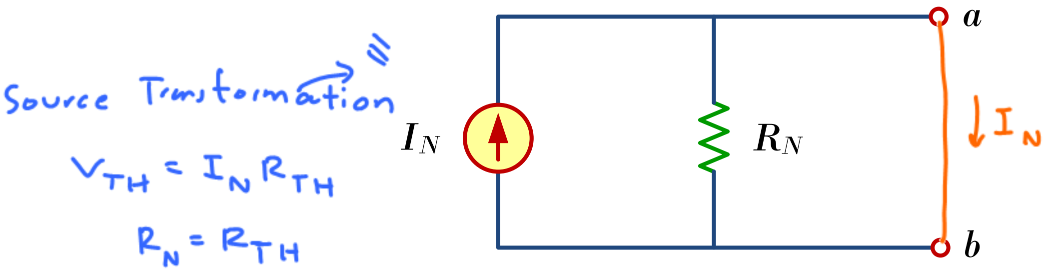
Norton's Theorem gives an alternative equivalent circuit to Thevenin's Theorem.

4.5.1. **Norton's Theorem:** A circuit can be replaced by an equivalent circuit consisting of a **current source** I_N in **parallel** with a **resistor** R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

Note: $R_N = R_{TH}$ and $I_N = \frac{V_{TH}}{R_{TH}}$. These relations are easily seen via source transformation.[4]



(a)



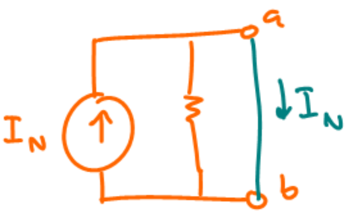
(b)

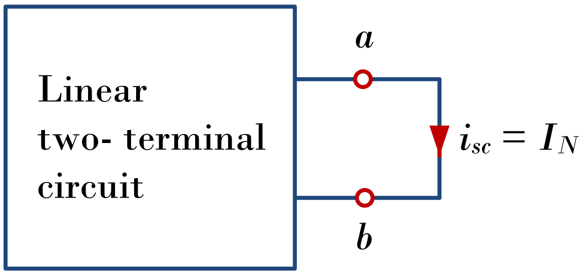
[4]For this reason, source transformation is often called Thevenin-Norton transformation.

Steps to Apply Norton's Theorem

S1: Find R_N (in the same way we find R_{TH}).

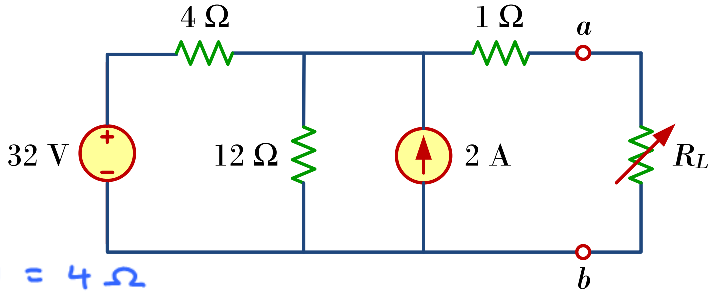
S2: Find I_N: Short circuit terminals a to b. I_N is the current passing through a and b.



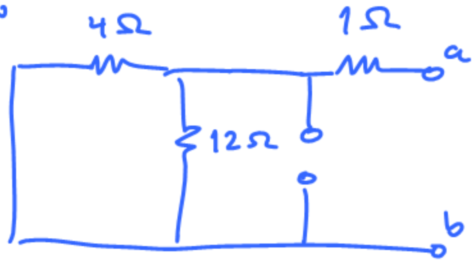


S3: Connect I_N and R_N in parallel.

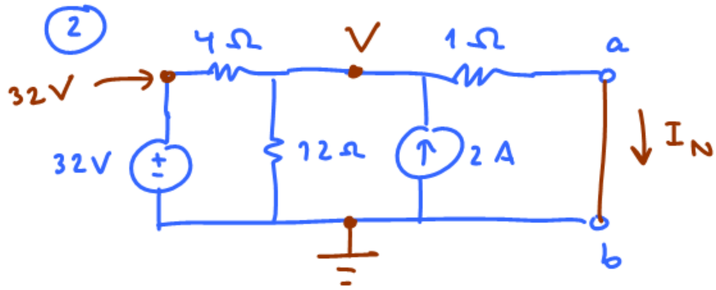
EXAMPLE 4.5.2. Back to the circuit in Example 4.4.5. Find the Norton equivalent circuit of the circuit shown below, to the left Directly of the terminals a-b.



① $R_N = (4/\!/12) + 1 = 4\ \Omega$



②

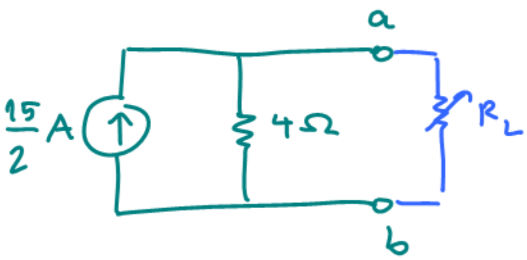


Nodal Analysis (KCL):

$$\frac{V-32}{4} + \frac{V-0}{12} + (-2) + \frac{V-0}{1} = 0$$

$$V = \frac{15}{2}\ V$$

$$I_N = \frac{V-0}{1} = \frac{15}{2}\ A$$



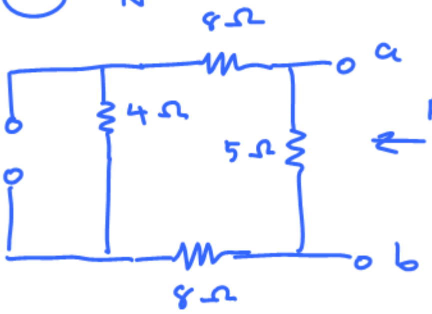
Previously, we have

$V_{TH} = 30\ V$

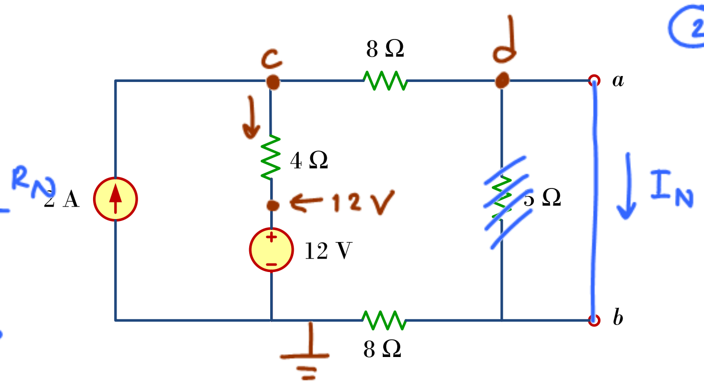
$R_{TH} = 4\ \Omega$

EXAMPLE 4.5.3. Find the Norton equivalent circuit of the circuit in the following figure at terminals a-b.

1 R_N



2



$$R_N = (8+4+8) // 5 = 20 // 5 = \frac{20 \times 5}{20+5} = 4\Omega$$

KCL @ c:

$$-2 + \frac{V_c - 12}{4} + \frac{V_c - V_d}{8} = 0$$

$\frac{V_c}{16}$

KCL @ d:

$$\frac{V_d - V_c}{8} + \frac{V_d - 0}{8} = 0$$

$$\frac{1}{4} V_d - \frac{V_c}{8} = 0$$

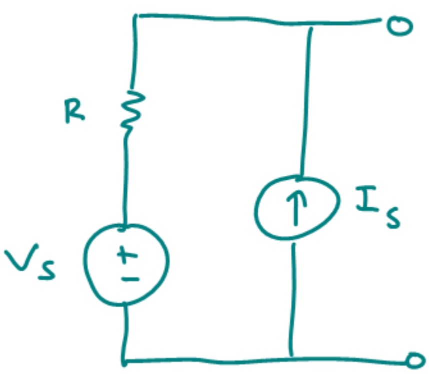
$$V_d = \frac{V_c}{2}$$

$$\frac{V_c - V_d}{8} = \frac{V_c - \frac{V_c}{2}}{8} = \frac{V_c}{16}$$

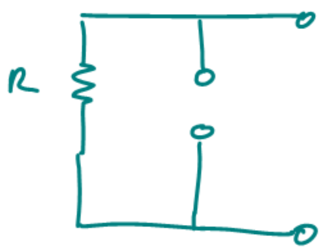
$$V_c = 16 V$$

$$I_N = \frac{V_c}{16} = 1 A$$

Extra Example :

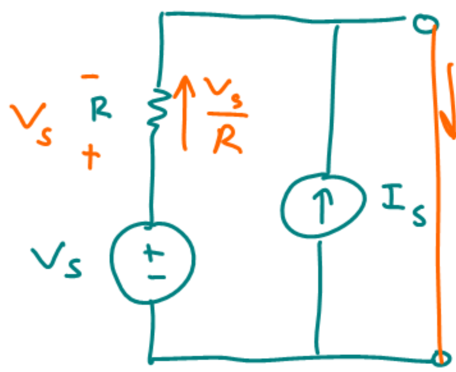


R_N



$R_N = R$

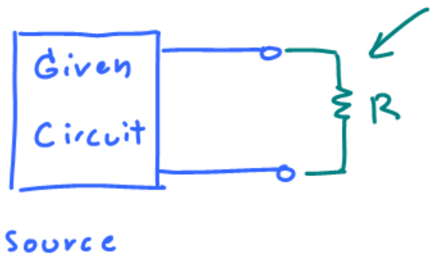
I_N



$$I_N = \frac{V_s}{R} + I_s$$

4.6. Maximum Power Transfer

In many practical situations, a circuit is designed to provide power to a load. In areas such as communications, it is desirable to maximize the power delivered to a load. We now address the problem of delivering the maximum power to a load when given a system with known internal losses.



The "load" (in this case the "load" is simply a resistor).

Goal: Find the value of R such that max power gets delivered to R.

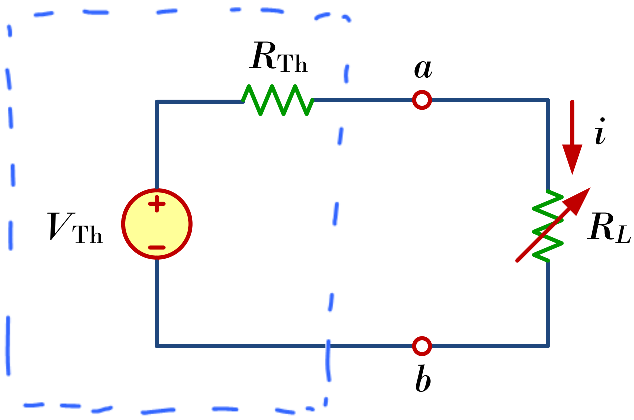
4.6.1. Questions:

(a) How much power can be transferred to the load under the most ideal conditions?
(b) What is the value of the load resistance that will absorb the maximum power from the source?

4.6.2. If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown below, the power delivered to the load resistor R_L is

$$p = i^2 R_L \quad \text{where} \quad i = \frac{V_{th}}{R_{th} + R_L}$$

$$= \left(\frac{V_{th}}{R_{th} + R_L}\right)^2 R_L$$



The derivative of p with respect to R_L is given by

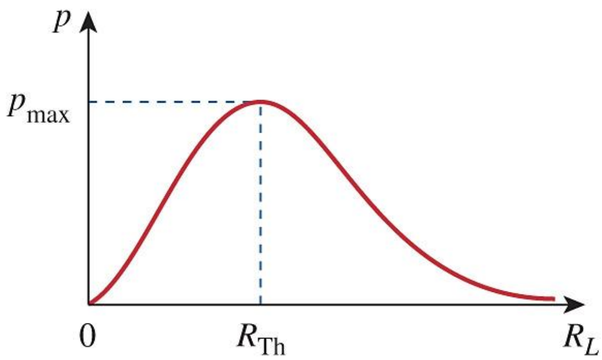
$$\begin{aligned}\frac{dp}{dR_L} &= 2i\frac{di}{dR_L}R_L + i^2\\ &= 2\frac{V_{th}}{R_{th}+R_L}\left(-\frac{V_{th}}{(R_{th}+R_L)^2}\right) + \left(\frac{V_{th}}{R_{th}+R_L}\right)^2\\ &= \left(\frac{V_{th}}{R_{th}+R_L}\right)^2\left(-\frac{2R_L}{R_{th}+R_L}+1\right).\end{aligned}$$

We then set this derivative equal to zero and get

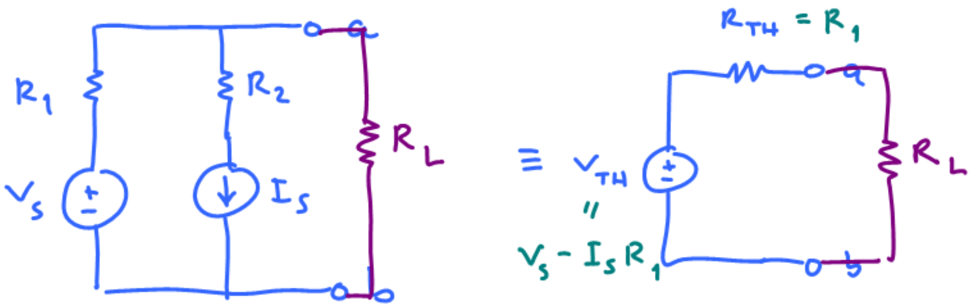
$$R_L = R_{TH}.$$

4.6.3. The maximum power transfer takes place when the load resistance R_L equals the Thevenin resistance R_{Th}. The corresponding maximum power transferred to R_L equals to

$$p_{\max} = \left(\frac{V_{th}}{R_{th}+R_{th}}\right)^2 R_{th} = \frac{V_{th}^2}{4R_{th}}.$$



EXAMPLE 4.6.4. Connect a load resistor R_L across the circuit in Example 4.4.4. Assume that $R_1 = R_2 = 14\Omega$, $V_s = 56$V, and $I_s = 2$A. Find the value of R_L for maximum power transfer and the corresponding maximum power.

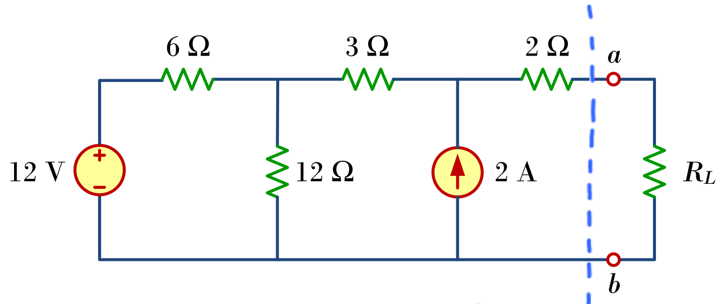


$$V_s - I_s R_1 = 56 - 2\times 14 = 28$$

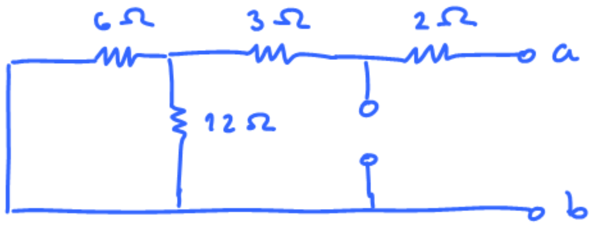
$$R_L^* = R_{TH} = R_1 = 14\Omega$$

$$P_{L,max} = \frac{V_{th}^2}{4R_{th}} = \frac{(V_s - R_1 I_s)^2}{4R_1} = \frac{28^2}{4\times 14} = 14\text{ W}$$

EXAMPLE 4.6.5. Find the value of R_L for maximum power transfer in the circuit below. Find the corresponding maximum power.



Find R_{TH}

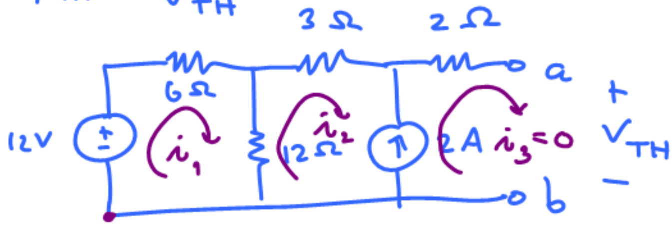


$$R_{TH} = \underbrace{(6 /\!/ 12)}_{\frac{6\times 12}{6+12} = 4} + 3 + 2 = 9\Omega$$

$$R_L^* = R_{TH} = 9\Omega$$

$$P_{L,max} = \frac{V_{TH}^2}{4R_{TH}} = \frac{22^2}{4\times 9} = 13.44\text{ W}$$

Find V_{TH}



Mesh 2: $i_2 = -2$A

Mesh 1:

$$+12 - i_1 \times 6 - (i_1 - i_2)\times 12 = 0$$

$$i_1 = -\frac{2}{3}$$

$$V_{TH} = 12 - 6i_1 - 3i_2 = 22\text{ V}$$

$= 12 - (-\frac{2}{3})6 - 3\times(-2)$

$= 12 + 4 + 6 = 22$

alternatively

$$= -12(i_2 - i_1) - 3i_2$$

$= -12(-2-(-\frac{2}{3})) - 3\times(-2) = -12(\frac{-4}{3}) + 6 = 16+6 = 22$