

DFT: Discrete Fourier Transform

Dr.Prapun Suksompong

Congruence (Integer modulo m)

- In this section, all letters stand for integers.
- $\gcd(n, m)$ = the greatest common divisor of n and m
- Let $d = \gcd(n, m)$
All the linear combinations $r \cdot n + s \cdot m$ of n and m are multiples of d .
- $a|b \Leftrightarrow a$ is a divisor of b .
- In an expression $\text{mod } m$, m is a strictly positive integer.
- $a = b \text{ mod } m \Leftrightarrow m | b - a$
- If $a = b \text{ mod } m$
 $c = d \text{ mod } m$
then
$$a + c = (b + d) \text{ mod } m$$
$$a - c = (b - d) \text{ mod } m$$
$$a \cdot c = (b \cdot d) \text{ mod } m$$
$$a^n = b^n \text{ mod } m \quad \text{with } n > 0$$
$$n \cdot a = (n \cdot b) \text{ mod } m$$
- $\langle m \rangle_N = m \text{ mod } N = m + rN$ for unique r that make $0 \leq m + rN < N$

N-point signal

- $x[n]$: **N-point signal** \Leftrightarrow
 - Has finite duration
 - Duration interval $\subset [0, N)$
- **Cyclic shift** of $x[n]$ by n_0 ; $0 \leq n < N$; $0 \leq n_0 < N$:

$$= x[\langle n - n_0 \rangle_N] = \begin{cases} x[n - n_0] & ; n_0 \leq n < N \\ x[N + n - n_0] & ; 0 \leq n < n_0 \\ 0 & ; \text{otherwise} \end{cases}$$

- $x[\langle n - n_0 \rangle_N]$ is another N-point signal
- Example

- $N = 5$

n	-2	-1	0	1	2	3	4	5	6	7
$x[n]$	0	0	a	b	c	d	e	0	0	0
$x[\langle n-2 \rangle_5]$	0	0	d	e	a	b	c	0	0	0

- **N-point circular convolution** of N-point signal

$$x_1[n] \circledast x_2[n] = \sum_{m=0}^{N-1} x_1[m] x_2[\langle n-m \rangle_N] = \sum_{m=0}^{N-1} x_1[\langle n-m \rangle_N] x_2[m]$$

Proof

$$\begin{aligned} \sum_{m=0}^{N-1} x_1[\langle n-m \rangle_N] x_2[m] &= \sum_{m=0}^n x_1[n-m] x_2[m] + \sum_{m=n+1}^{N-1} x_1[N+n-m] x_2[m] \\ &= \sum_{\ell=0}^n x_1[\ell] x_2[n-\ell] + \sum_{\ell=n+1}^{N-1} x_1[\ell] x_2[n+N-\ell] \\ &= \sum_{\ell=0}^n x_1[\ell] x_2[\langle n-\ell \rangle_N] + \sum_{\ell=n+1}^{N-1} x_1[\ell] x_2[\langle n-\ell \rangle_N] \end{aligned}$$

- If $x_1[m]$ has many 0's, use $\sum_{m=0}^{N-1} x_1[m] x_2[\langle n-m \rangle_N]$, eliminating each of $x_2[\langle n-m \rangle_N]$

that is multiplied by $x_1[m] = 0$

- Example:

- $N = 3$

Let $y[n] = x_1[n] \circledast x_2[n]$

$$y[n] = \sum_{m=0}^2 x_1[m] x_2[\langle n-m \rangle_3] = x_1[0] x_2[n] + x_1[1] x_2[\langle n-1 \rangle_3] + x_1[2] x_2[\langle n-2 \rangle_3]$$

$$y[0] = x_1[0] x_2[0] + x_1[1] x_2[\langle -1 \rangle_3] + x_1[2] x_2[\langle -2 \rangle_3]$$

$$= x_1[0] x_2[0] + x_1[1] x_2[2] + x_1[2] x_2[1]$$

$$y[1] = x_1[0] x_2[1] + x_1[1] x_2[\langle 0 \rangle_3] + x_1[2] x_2[\langle -1 \rangle_3]$$

$$= x_1[0] x_2[1] + x_1[1] x_2[0] + x_1[2] x_2[2]$$

$$y[2] = x_1[0] x_2[2] + x_1[1] x_2[\langle 1 \rangle_3] + x_1[2] x_2[\langle 0 \rangle_3]$$

$$= x_1[0] x_2[2] + x_1[1] x_2[1] + x_1[2] x_2[0]$$

- To find $x_1[n] \circledast x_2[n]$ using circular convolution rule

- $\hat{X}_1[k] = x_1[0] + x_1[1] \psi_3^{-k} + x_1[2] \psi_3^{-2k} = x_1[0] + x_1[1] A + x_1[2] A^2$

$$\hat{X}_2[k] = x_2[0] + x_2[1]\psi_3^{-k} + x_2[2]\psi_3^{-2k} = x_2[0] + x_2[1]A + x_2[2]A^2$$

The multiplication $\hat{X}_1[k] \cdot \hat{X}_2[k]$ can be easily find with TI calculator or Mathcad (select A, then choose Symbolics > Polynomial Coefficients):

$$\hat{X}_1[k] \cdot \hat{X}_2[k] = \begin{bmatrix} 1 & A & A^2 & A^3 & A^4 \end{bmatrix} \begin{bmatrix} x_{10}x_{20} \\ x_{10}x_{21} + x_{11}x_{20} \\ x_{10}x_{22} + x_{11}x_{21} + x_{12}x_{20} \\ x_{11}x_{22} + x_{12}x_{21} \\ x_{12}x_{22} \end{bmatrix}$$

But $A^3 = 1, A^4 = A$; therefore,

$$\hat{X}_1[k] \cdot \hat{X}_2[k] = \begin{bmatrix} 1 & A & A^2 \end{bmatrix} \begin{bmatrix} x_{10}x_{20} + x_{11}x_{22} + x_{12}x_{21} \\ x_{10}x_{21} + x_{11}x_{20} + x_{12}x_{22} \\ x_{10}x_{22} + x_{11}x_{21} + x_{12}x_{20} \end{bmatrix}.$$

Obviously (actually from inverse DFT),
$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix} = \begin{bmatrix} x_{10}x_{20} + x_{11}x_{22} + x_{12}x_{21} \\ x_{10}x_{21} + x_{11}x_{20} + x_{12}x_{22} \\ x_{10}x_{22} + x_{11}x_{21} + x_{12}x_{20} \end{bmatrix}.$$

• Example:

• $x_1[n] = 2 \ 3 \ 1 \ 1, x_2[n] = 1 \ 0 \ 1 \ 2$

$$\begin{aligned} \hat{X}_1[k] \cdot \hat{X}_2[k] &= (2 + 3\psi_4^{-k} + 1\psi_4^{-2k} + 1\psi_4^{-3k}) \cdot (1 + 0\psi_4^{-k} + 1\psi_4^{-2k} + 2\psi_4^{-3k}) \\ &= 2 + 3\psi_4^{-k} + 3\psi_4^{-2k} + 8\psi_4^{-3k} + 8\psi_4^{-4k} + 3\psi_4^{-5k} + 2\psi_4^{-6k} \\ &= 10 + 6\psi_4^{-k} + 5\psi_4^{-2k} + 8\psi_4^{-3k} \end{aligned}$$

$$x_1[n] \otimes x_2[n] = 10 \ 6 \ 5 \ 8$$

DFT

- $\psi_N = e^{j\frac{2\pi}{N}}$
- $(\psi_N)^N = 1$
- $\overline{\psi_N} = (\psi_N)^{-1}$
- $N \bmod p = 0 \Rightarrow \psi_N^p = \psi_{\frac{N}{p}}$
- $\psi_{kN}^N = \psi_k$
- For even n, $\psi_N^{\frac{N}{2}} = e^{j\pi} = -1$

$$\bullet \sum_{n=0}^{N-1} \psi_N^{-kn} = \begin{cases} N & k = mN \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Proof } (\psi_N)^N = \left(e^{j\frac{2\pi}{N}} \right)^N = e^{j2\pi} = 1$$

$$\text{Proof } \overline{\psi_N} = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{N}} = \left(e^{j\frac{2\pi}{N}} \right)^{-1}$$

$$\text{Proof } N \bmod p = 0 \Rightarrow \psi_N^p = \left(e^{j\frac{2\pi}{N}} \right)^p = e^{j\frac{2\pi}{N/p}} = \psi_{N/p}$$

$$\text{Proof } \sum_{n=0}^{N-1} e^{j2\pi n \frac{k}{N}} = \begin{cases} N & \text{if } \frac{k}{N} \in I \\ 0 & \text{if } \frac{k}{N} \notin I \end{cases}$$

$$\sum_{n=0}^{N-1} \psi_N^{-kn} = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(-kn)} = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk} = \begin{cases} N & \text{if } \frac{k}{N} \in I \\ 0 & \text{if } \frac{k}{N} \notin I \end{cases}$$

$$\bullet [\Psi_N]_{pq} = \psi_N^{-(p-1)(q-1)} ; 1 \leq p, q \leq N$$

$$\bullet \Psi_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \psi_N^{-1} & \psi_N^{-2} & \dots & \psi_N^{-(N-1)} \\ 1 & \psi_N^{-2} & \psi_N^{-4} & \dots & \psi_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi_N^{-(N-1)} & \psi_N^{-2(N-1)} & \dots & \psi_N^{-(N-1)(N-1)} \end{bmatrix}$$

$$\bullet \Psi_N^{-1} = \frac{1}{N} \overline{\Psi_N}$$

Proof

$$\begin{aligned}
\left[\overline{\Psi_N} \Psi_N \right]_{pq} &= \sum_{\ell=1}^N \left[\overline{\Psi_N} \right]_{p\ell} \left[\Psi_N \right]_{\ell q} = \sum_{\ell=1}^N \psi_N^{(p-1)(\ell-1)} \psi_N^{-(\ell-1)(q-1)} \\
&= \sum_{\ell=1}^N \psi_N^{(\ell-1)(p-q)} = \sum_{m=0}^{N-1} \psi_N^{m(p-q)} \\
&= \begin{cases} N & ; p = q \Rightarrow \psi_N^{p-q} = 1 \\ \frac{1 - \psi_N^{N(p-q)}}{1 - \psi_N^{(p-q)}} = 0 & ; p \neq q \end{cases} \\
&= NI_N
\end{aligned}$$

- **N-point DFT** of the N-point signal $x[n]$:

$$\hat{X}[k] = \hat{X}\left(\omega = k \frac{2\pi}{N}\right) = \sum_{n=0}^{N-1} x[n] \psi_N^{-nk} ; 0 \leq k < N$$

- $\hat{X}[k] = \hat{X}\left(\omega = k \frac{2\pi}{N}\right) = \sum_{n=0}^{N-1} x[n] e^{-jnk \frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[n] \psi_N^{-nk} ; 0 \leq k < N$

- $$\begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \\ \hat{X}[2] \\ \vdots \\ \hat{X}[n] \end{bmatrix} = \Psi_N \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \psi_N^{-1} & \psi_N^{-2} & \cdots & \psi_N^{-(N-1)} \\ 1 & \psi_N^{-2} & \psi_N^{-4} & \cdots & \psi_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi_N^{-(N-1)} & \psi_N^{-2(N-1)} & \cdots & \psi_N^{-(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[n] \end{bmatrix}$$

- $\delta[n] \xleftrightarrow{DFT} \sum_{n=0}^{N-1} \delta[n] \psi_N^{-nk} = \psi_N^{-0k} = 1$

- $(DFT^{-1}) x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}[k] \psi_N^{nk} \xleftrightarrow{DFT} \hat{X}[k] = \sum_{n=0}^{N-1} x[n] \psi_N^{-nk} \quad (DFT)$

Proof $\underline{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[n] \end{bmatrix} ; \underline{\hat{X}} = \begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \\ \hat{X}[2] \\ \vdots \\ \hat{X}[n] \end{bmatrix}$

$$\Rightarrow \underline{\hat{X}} = \Psi_N \underline{x} \Rightarrow \underline{x} = \frac{1}{N} \Psi_N^* \underline{\hat{X}}$$

- Given tool for computing DFT

$$\hat{X}[k] \xrightarrow{0 \leq k < N} \boxed{\text{DFT}} \rightarrow \boxed{\text{Time reverse}} \rightarrow \boxed{\times \frac{1}{N}} \xrightarrow{0 \leq k < N} x[n]$$

DFT^{-1}

Proof After DFT-finder $g[q] = \sum_{p=0}^{N-1} \hat{X}[p] \psi_N^{-pq}$,

Time reverse: $g[N-q] = \sum_{p=0}^{N-1} \hat{X}[p] \psi_N^{-p(N-q)} = \sum_{p=0}^{N-1} \hat{X}[p] \psi_N^{pq}$

- $\hat{X}[k] \xrightarrow{DFT^{-1}} x[n] \xrightarrow{DTFT} \hat{X}(\omega)$
- $x[\langle n - n_0 \rangle_N] \xleftrightarrow{DFT} \psi_N^{-kn_0} \hat{X}[k]$
 $0 \leq n < N \qquad \qquad \qquad 0 \leq k < N$

Proof

$$\begin{aligned} \sum_{n=0}^{N-1} x[\langle n - n_0 \rangle_N] \psi_N^{-nk} &= \sum_{n=n_0}^{N-1} x[n - n_0] \psi_N^{-nk} + \sum_{n=0}^{n_0-1} x[N + n - n_0] \psi_N^{-nk} \\ &= \sum_{\ell=0}^{N-n_0-1} x[\ell] \psi_N^{-(\ell+n_0)k} + \sum_{\ell=N-n_0}^{N-1} x[\ell] \psi_N^{-(\ell+n_0-N)k} \\ &= \sum_{\ell=0}^{N-1} x[\ell] \psi_N^{-(\ell+n_0)k} \end{aligned}$$

- $\delta[n - n_0] \xleftrightarrow{DFT} \psi_N^{-n_0k} ; 0 \leq n_0 < N$

Proof $\delta[\langle n - n_0 \rangle_N] = \begin{cases} \delta[n - n_0] & ; n_0 \leq n < N \\ \delta[N + n - n_0] & ; 0 \leq n < n_0 \\ 0 & ; \text{otherwise} \end{cases} = \delta[n - n_0]$

$$\delta[\langle n - n_0 \rangle_N] \xrightarrow{DFT} \psi_N^{-kn_0} \times 1$$

$0 \leq n < N \qquad \qquad \qquad 0 \leq k < N$

• **Circular convolution rule:**

$$x_1[n] \circledast x_2[n] \xleftrightarrow{DFT} \hat{X}_1[k] \cdot \hat{X}_2[k]$$

Proof

$$\begin{aligned} \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x_1[m] x_2[\langle n - m \rangle_N] \right) \psi_N^{-nk} &= \sum_{m=0}^{N-1} x_1[m] \left(\sum_{n=0}^{N-1} x_2[\langle n - m \rangle_N] \psi_N^{-nk} \right) \\ &= \sum_{m=0}^{N-1} x_1[m] \left(\hat{X}_2[k] \psi_N^{-mk} \right) \end{aligned}$$

- To use DFT to compute regular convolutions of time-limited (finite-duration) signal:

If $x_1[n]$ has duration interval $0 \leq n < N_1$

$x_2[n]$ has duration interval $0 \leq n < N_2$

Let $N = N_1 + N_2 - 1$,

then $x_1[n] * x_2[n] = x_1[n] \circledast x_2[n]$ for $0 \leq n < N$ and = 0 otherwise

To see this,

$x_1[n] * x_2[n]$ has finite duration at most $N = N_1 + N_2 - 1$

Think of $x_1[n]$ and $x_2[n]$ as N -point signal whose last values are 0 (0-padding)

- Block convolution**

$h[n]$ has duration interval $0 \leq n < P$ (impulse response of a causal FIR system)

$w[n]$ = a (possibly) infinite-duration signal

To compute $h[n] * w[n]$

- Divide $w[n]$ into blocks of some specific length L (typically $L \gg P$)

$$w_r[n] = \begin{cases} w[rL + n] & ; 0 \leq n < L \\ 0 & ; \text{otherwise} \end{cases} \quad \text{duration interval } 0 \leq n < L$$

$$w[n] = \sum_{r=-\infty}^{\infty} w_r[n - rL], \quad \forall n$$

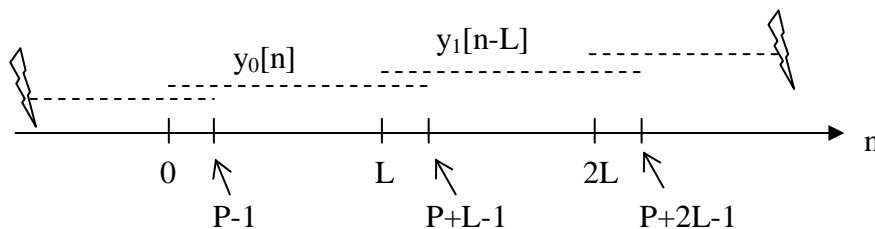
- $y_r[n] = h[n] * w_r[n]$ duration interval $0 \leq n < P + L - 1$

Find $y_r[n]$ by $(P+L-1)$ -point circular convolution

- $h[n] * w[n] = \sum_{r=-\infty}^{\infty} y_r[n - rL], \quad \forall n$

$$\begin{aligned} \text{Proof } h[n] * w[n] &= h[n] * \sum_{r=-\infty}^{\infty} w_r[n - rL] = \sum_{r=-\infty}^{\infty} h[n] * w_r[n - rL] \\ &= \sum_{r=-\infty}^{\infty} y_r[n - rL] \end{aligned}$$

Note that $h[n] * w_r[n - rL] = y_r[n - rL]$ from the time-invariance property of convolution.



- **Frequency-sampling approximation** of $h[n]$

To approximate $h[n]$:

- Sampling $\hat{H}(\omega)$ at $\omega = k \frac{2\pi}{N}$; $0 \leq k < N$ by

$$e^{jk \frac{2\pi}{N} n} \xrightarrow{h[n]} \text{yield } \tilde{H}[k] = \hat{H}\left(\omega = k \frac{2\pi}{N}\right)$$

- Use (DFT)⁻¹, $\tilde{h}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{H}[k] \psi_N^{nk} = \sum_{r=-\infty}^{\infty} h[n - rN]$; $0 \leq n < N$

$$\frac{(\hat{H}(\omega)|_{\omega=\omega_0}) e^{jn\omega_0}}{e^{jn\omega_0}} = (\hat{H}(\omega)|_{\omega=\omega_0})$$

$$\tilde{H}[k] = \hat{H}\left(\omega = k \frac{2\pi}{N}\right) \xrightarrow{DFT^{-1}} \tilde{h}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} h[n - rN] & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

- $y[n] = \sum_{r=-\infty}^{\infty} x[n - rN] \xleftrightarrow{DFT} \hat{Y}[k] = \hat{X}\left(\omega = k \frac{2\pi}{N}\right)$

Proof

$$\begin{aligned} \tilde{h}[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{H}[k] \psi_N^{nk} = \frac{1}{N} \sum_{k=0}^{N-1} \hat{H}\left(k \frac{2\pi}{N}\right) \psi_N^{nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=-\infty}^{\infty} h[m] e^{-jmk \frac{2\pi}{N}} \right) e^{j \frac{2\pi}{N} nk} \\ &= \sum_{m=-\infty}^{\infty} h[m] \left(\frac{1}{N} \sum_{k=0}^{N-1} e^{j \frac{2\pi}{N} k(n-m)} \right) \end{aligned}$$

$$\frac{1}{M} \sum_{\ell=0}^{M-1} e^{j2\pi \ell \frac{n}{M}} = \begin{cases} 1 ; & \text{if } \frac{n}{M} \in I \\ 0 ; & \text{if } \frac{n}{M} \notin I \end{cases} \Rightarrow \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi k \frac{n-m}{N}} = \begin{cases} 1 ; & \text{if } \frac{n-m}{N} \in I \\ 0 ; & \text{if } \frac{n-m}{N} \notin I \end{cases}$$

$$\tilde{h}[n] = \sum_{\substack{m=-\infty \\ \frac{n-m}{N} \in I}}^{\infty} h[m]$$

So, the summation only include m of the form $m = n - rN$; $r \in I$

$$\text{Thus, } \tilde{h}[n] = \sum_{r=-\infty}^{\infty} h[n - rN]$$

$$\text{Proof 2 : } \tilde{h}[n] = \sum_{r=-\infty}^{\infty} h[n - rN] \xrightarrow{DFT} \tilde{H}[k] = \hat{H}\left(\omega = k \frac{2\pi}{N}\right)$$

$$\text{First, note that } \hat{H}\left(k \frac{2\pi}{N}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-jnk \frac{2\pi}{N}} = \sum_{n=-\infty}^{\infty} h[n] \psi_N^{-nk}.$$

$$\begin{aligned} \tilde{H}[k] &= \sum_{n=0}^{N-1} \tilde{h}[n] \psi_N^{-nk} = \sum_{n=0}^{N-1} \left(\sum_{r=-\infty}^{\infty} h[n + rN] \right) \psi_N^{-nk} \\ &= \sum_{r=-\infty}^{\infty} \sum_{n=0}^{N-1} (h[n + rN] \psi_N^{-nk}) \end{aligned}$$

$$\text{Substitute } \ell = n + rN \Rightarrow n = \ell - rN$$

$$\tilde{H}[k] = \sum_{r=-\infty}^{\infty} \sum_{\ell=rN}^{(r+1)N-1} (h[\ell] \psi_N^{-(\ell-rN)k}) = \sum_{r=-\infty}^{\infty} \sum_{\ell=rN}^{(r+1)N-1} (h[\ell] \psi_N^{-\ell k})$$

$$\text{Note that } \sum_{r=-\infty}^{\infty} \sum_{\ell=rN}^{(r+1)N-1} x(\ell) = \sum_{r=-\infty}^{\infty} \sum_{\ell=rN}^{rN+(N-1)} x(\ell) = \sum_{\ell=-\infty}^{\infty} x[\ell].$$

$$\text{Thus, } \tilde{H}[k] = \sum_{\ell=-\infty}^{\infty} h[\ell] \psi_N^{-\ell k} = \hat{H}\left(k \frac{2\pi}{N}\right).$$

- Time-aliasing

- If $h[n]$ has duration interval contained in $0 \leq n < N$,

$$\tilde{h}[n] = h[n]; 0 \leq n < N$$

- Example: if $h[n]$ is a $(2N)$ -point signal

$$\begin{aligned}
\hat{H}\left(k\frac{2\pi}{N}\right) &= \sum_{n=-\infty}^{\infty} h[n]e^{-jnk\frac{2\pi}{N}} = \sum_{n=-\infty}^{\infty} h[n]\psi_N^{-nk} = \sum_{n=0}^{2N-1} h[n]\psi_N^{-nk} \\
&= \sum_{n=0}^{N-1} h[n]\psi_N^{-nk} + \sum_{n=N}^{2N-1} h[n]\psi_N^{-nk} \\
&= \sum_{n=0}^{N-1} h[n]\psi_N^{-nk} + \sum_{\ell=0}^{N-1} h[\ell+N]\psi_N^{-(\ell+N)k} \\
&= \sum_{n=0}^{N-1} h[n]\psi_N^{-nk} + \sum_{\ell=0}^{N-1} h[\ell+N]\psi_N^{-\ell k} \\
&= \sum_{n=0}^{N-1} (h[n] + h[n+N])\psi_N^{-nk}
\end{aligned}$$

$$\tilde{h}[n] = h[n] + h[n+N]; 0 \leq n < N$$

So, $\tilde{h}[n] = h[n]; 0 \leq n < N$ iff $h[n] = 0$ for $n \geq N$ (no folding)

- **Windowing a signal** to get an approximation of $\hat{X}(\omega)$

Given $x[n] -\infty < n < \infty$

$$\text{Look at } y[n] = x[n]p_L[n] = \begin{cases} x[n] & ; -L < n < L \\ 0 & ; |n| \geq L \end{cases} \quad (\text{length} = 2L-1 = N)$$

$$\text{Find } \hat{Y}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin\left(\left(L - \frac{1}{2}\right)\mu\right)}{\sin\left(\frac{\mu}{2}\right)} \hat{X}(\omega - \mu) d\mu$$

$$\bullet \quad p_L[n] = \begin{cases} 1 & ; -L < n < L \\ 0 & ; |n| \geq L \end{cases} \xleftrightarrow{DTFT} \frac{\sin\left(\left(L - \frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

- Zeros @ $\frac{k\pi}{L - \frac{1}{2}}$
- = $2L-1$ @ $\omega \rightarrow 0$

$$\bullet \quad \text{Need width of } \hat{X}(\omega) \text{'s feature} > \frac{2\pi}{L - \frac{1}{2}} = \frac{4\pi}{N}$$

- If $L \rightarrow \infty$, $\lim_{L \rightarrow \infty} p_L[n] = 1 \xleftrightarrow{DTFT} 2\pi\delta(\omega)$, and $\hat{Y}(\omega) \rightarrow \hat{X}(\omega)$

- Lots of “sharp activity” in $\hat{X}(\omega) \Rightarrow$ need bigger N
- Given $x[n]; 0 \leq n < N$

Can use M -point DFT's to find $\hat{X}[k]$ only for $k = \frac{N}{M} \ell = r\ell; 0 \leq \ell < M; r = \frac{N}{M} \in I$

by let $y[n] = \sum_{p=0}^{r-1} x[n + (p-1)M]$, then $\hat{X}[k] = \hat{Y}[\ell]$

Proof

$$\begin{aligned} \hat{X}[k] &= \sum_{n=0}^{N-1} x[n] \psi_N^{-nk} = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} \left(\frac{N}{M} \ell\right) n} = \sum_{n=0}^{N-1} x[n] \psi_M^{-\ell n} \\ &= \sum_{p=0}^{r-1} \sum_{n=pM}^{(p+1)M-1} x[n] \psi_M^{-\ell n} = \sum_{p=0}^{r-1} \sum_{n=0}^{M-1} x[n + (p-1)M] \psi_M^{-\ell(n+(p-1)M)} \\ &= \sum_{n=0}^{M-1} \sum_{p=0}^{r-1} x[n + (p-1)M] \psi_M^{-\ell n} \end{aligned}$$

- Let $x[n]$ be a signal of duration N .

If already have tool for M -point DFT's.

Can find $\hat{X}(\omega)$ for $\omega_k = \omega_0 + k \frac{2\pi}{M}$, by constructing $y[n]$ so $\hat{Y}[k] = \hat{X}(\omega_k)$.

To do this,

- Let $q[n] = e^{-j\omega_0 n} x[n]$, then $\hat{Q}(\omega) = \hat{X}(\omega + \omega_0)$ and

$$\hat{Q}\left(k \frac{2\pi}{M}\right) = \hat{X}\left(k \frac{2\pi}{M} + \omega_0\right) = \hat{X}(\omega_k)$$

- Want $\hat{Y}[k] = \hat{Q}\left(k \frac{2\pi}{M}\right)$,

From $y[n] = \sum_{r=-\infty}^{\infty} x[n - rN] \xrightarrow{DFT} \hat{Y}[k] = \hat{X}\left(\omega = k \frac{2\pi}{N}\right)$, so need

$$y[n] = \begin{cases} \sum_{r=-\infty}^{\infty} q[n + rM] & 0 \leq n < M \\ 0 & \text{otherwise} \end{cases}$$

- Note that not all r 's are used. only ones that satisfy

$$0 < n+rM < N \Rightarrow -\frac{n}{M} \leq r < \frac{N-n}{M}, \text{ but } 0 \leq n < M. \text{ So, } -1 < r < \frac{N}{M}$$

$$\Rightarrow 0 \leq r < \frac{N}{M}$$