Sirindhorn International Institute of Technology
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MAS 116: Mathematics I

Semester: 3/2008
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Disclaimer: The original version of this note is created by Asst. Prof. Dr. Nirattaya Khamsemanan. This is a revised and expanded version. The content is based on the textbook by Anton, Bivens, and Davis. The note should be used as a supplementary material and should not be considered as a replacement for the main textbook. Of course, there might be some typos and/or mistakes.

Contents

1 Functions

Definition 1.1. If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that y **is a function of** x.

- In 1673, Leibniz coined the term function to indicate the dependence of one quantity on another.
- Four common methods for representing functions are
  (a) Numerically by tables
  (b) Geometrically by graphs
  (c) Algebraically by formulas
  (d) Verbally

Definition 1.2. A **function** f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$ (read "f of x").

- The term *unique* means "exactly one". Thus, a function cannot assign two different outputs to the same input.

Definition 1.3. For a given input x, the output of a function f is called the **value** of f at x or the **image** of x under f. Sometimes we will want to denote the output by a single letter say y, and write

$$y = f(x).$$

This equation express y as a function of x; the variable x is called the **independent variable** (or **argument**) of f, and the variable y is called the **dependent variable** of f.

- This terminology is intended to suggest that x is free to very, but once x has a specific value, a corresponding value of y is determined.
- A function in which the independent and dependent variables are real numbers is called a **real-valued function of a real variable**.

Example 1.4. Let $f(x) = \frac{x-2}{x+1}$.

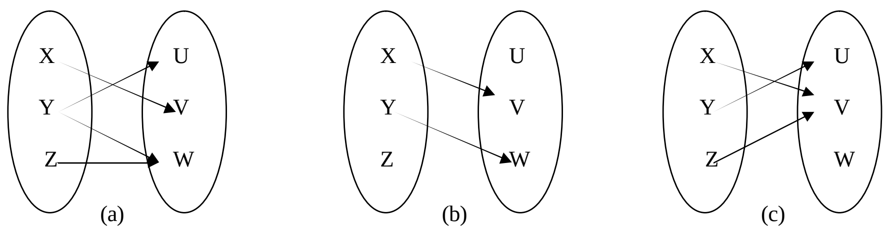
(a) Compute $f(2)$
Solution:

(b) Compute $f(x+1)$
Solution:

1.5. If x and y are related by the equation $y = f(x)$, then the set of all allowable inputs (x-values) is called the **domain** of f, and the set of outputs (y-values) that result when x varies over the domain is called the **range** of f.

- Sometimes physical or geometric considerations impose restrictions on the allowable inputs of a function.
- When a function is defined by a mathematical formula, the formula itself may impose restrictions on the allowable inputs.
    - Division by zero is undefined.
    - If we consider only real-valued functions of a real variable, square roots of negative values (which produces imaginary values) are not allowed.
    - TBD: The natural log function ($f(x) = \ln(x)$) only accepts positive inputs.
- The domain and range of a function f can be pictured by projecting the graph of $y = f(x)$ onto the coordinate axes.

Example 1.6. : Which of the followings is a function whose domain is $\{X, Y, Z\}$.



Solution:

Definition 1.7. If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the **natural domain** of the function.

- Algebraic expressions are frequently simplified by canceling common factors in the numerator and denominator. This process can alter the natural domain. If the domain must be preserved, then one must impose the restrictions on the simplified function explicitly. For example, if we wanted to preserve the natural domain of the function

$$f(x) = \frac{x^2 - 4}{x - 2},$$

then we have to express the simplified form of the function as

$$f(x) = x + 2,\ x \neq 2.$$

Example 1.8. : Find the (natural) domains of the following functions.

(a) $f(x) = 3x + 2$
Solution:

(b) $f(x) = \sqrt{3x + 2}$
Solution:

(c) $f(x) = \frac{1}{3x+2}$
Solution:

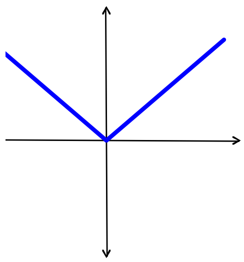
(d) $f(x) = \frac{x\sqrt{x}}{x-1}$
Solution:

(e) $f(x) = x + \frac{1}{\sqrt{x}}$
Solution:

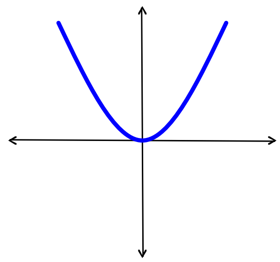
1.1 Graphs of Functions

If f is a real-valued function of a real variable, then the graph of f in the xy-plane is defined to be the graph of the equation $y = f(x)$.

Example 1.9. Graphs of six basic functions



(a)



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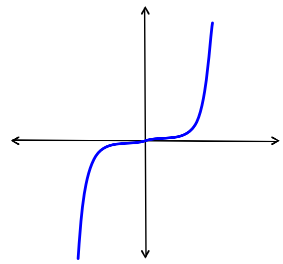

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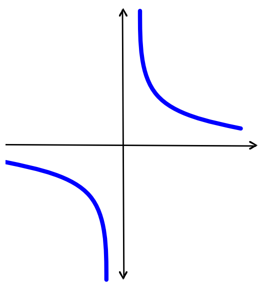
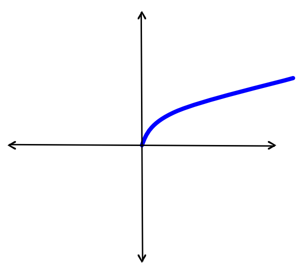
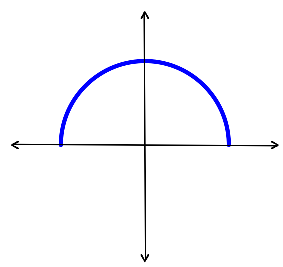


(d)



(e)



(f)

- (a) represents the function $f(x) = |x|$

  Domain: .... Range: ....

- (b) represents the function $f(x) = x^2$

  Domain: .... Range: ....

- (c) represents the function $f(x) = x^3$

  Domain: .... Range: ....

- (d) represents the function $f(x) = 1/x$

  Domain: .... Range: ....

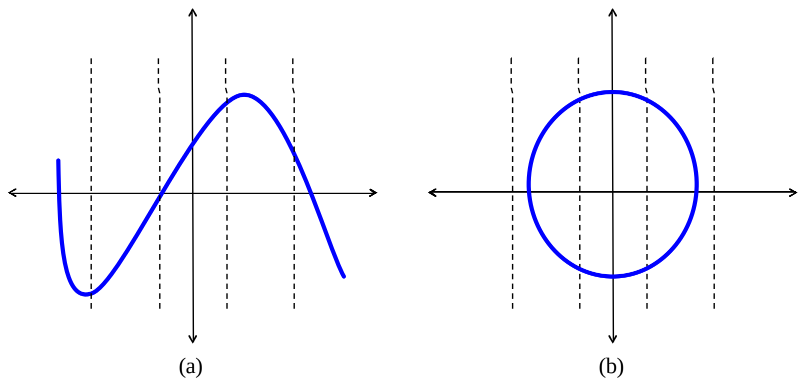
- (e) represents the function $f(x) = \sqrt{x}$

  Domain: .... Range: ....

- (f) represents the function $f(x) = \sqrt{1 - x^2}$

  Domain: .... Range: ....

1.10. The Vertical Line Test: A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.



- In (a), each vertical line intersect the graph only once. Thus, the graph in (a) represents a function.
- In (b), there are some vertical lines that intersect the graph twice. Thus this graph does not represent a function.

1.11. For a function that is defined **piecewise**, the formula for f changes, depending on the value of x. The points where the formula for f changes are called the **breakpoints**.

Example 1.12. Let

$$f(x) = \begin{cases} |x| & \text{if } x \leq 0; \\ x^2 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of f.
Solution:

(b) Find $f(-1), f(0), f(1)$
Solution:

Example 1.13. Sketch the graph of

$$f(x) = \begin{cases} 0, & x \leq -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \geq 1. \end{cases}$$

Solution:

1.2 Some Elementary Functions

Definition 1.14. A **power function** is a function of the form

$$f(x) = kx^r$$

where k and r are any real numbers.

Example 1.15. Some examples of power functions.

Definition 1.16. **A polynomial function** of degree n is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, (a_n \neq 0)$$

where $a_n, a_{n-1}, ..., a_1, a_0$ are constants and n is non-negative integer. The domain is all real numbers. The coefficient a_n is called the leading coefficient.

Example 1.17. Some examples of polynomial functions.
Solution:

Definition 1.18. A function that can be expressed as a ratio of two polynomials is called a **rational function**.

- If $P(x)$ and $Q(x)$ are polynomials, then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

consists of all values of x such that $Q(x) \neq 0$.

Example 1.19. Some examples of rational functions.
Solution:

Definition 1.20. Functions that can be constructed from polynomials by applying finitely many algebraic operations (addition, subtraction, multiplication, division, and root extraction) are called **algebraic functions**.

Definition 1.21. A function of the form $f(x) = b^x$, where $b > 0$, is called **an exponential function with base** b.

- Some examples are

$$f(x) = 2^x, \quad f(x) = \left(\frac{1}{2}\right)^x, \quad f(x) = \pi^x.$$

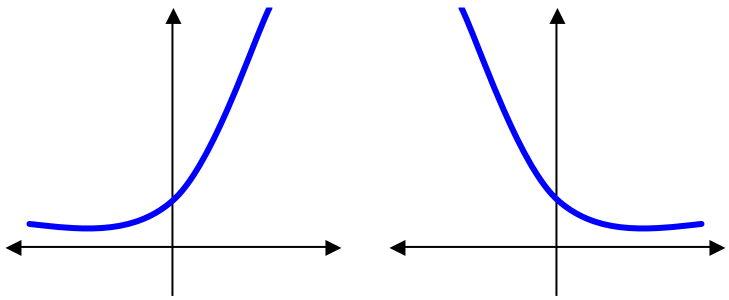
- Note that an exponential function has a constant base and variable exponent.
- In many applications, function of the form $y = ab^x$, where a, b, k are constants with $b \neq 1$ are also call exponential function.

1.22. The graph of b^x has one of three general forms, depending on the value of b

(a) If $b > 1$, the function b^x is increasing and it approaches the x-axis as x become large.

(b) If $0 < b < 1$, the function b^x is decreasing and it approaches the x-axis as x becomes negatively large.

(c) If $b = 1$, the function b^x is a constant.



Domain:.................................. Range:...............................

y-intercept:.................. x-intercept:.................. Asymptote:.......................

1.23. Properties of Exponential functions

(a) The function $f(x) = b^x$ is defined for all real values of x, so its natural domain is $(-\infty, +\infty)$.

- When $b \neq 1$, its range is $(0, \infty)$.

(b) TBD: The function $f(x) = b^x$ is continuous.

(c) If $b^x = b^y$, then $x = y$

(d) $(b^x)(b^y) = b^{x+y}$.

(e) $\frac{b^x}{b^y} = b^{x-y}$.

(f) $(b^x)^y = (b^y)^x = b^{xy}$.

(g) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

(h) $\frac{1}{b^x} = b^{-x}$.

(i) $\sqrt[n]{b} = b^{1/n}$

(j) $b^0 = 1$

Definition 1.24. If $b > 0$ and $b \neq 1$, then for a positive value of x the expression

$$\log_b x$$

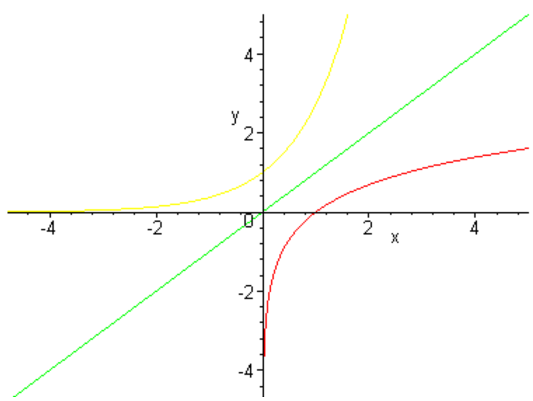
denotes that exponent to which b must be raised to produce x.

- Read "the logarithm to the base b of x" or "log base b of x" or the "log of x base b"
- Logarithms with base e are called **natural logarithm**.
    - It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x").
- If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.
    - The graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$.

Example 1.25. .

Exponential form of equation	Logarithmic form of equation
$9 = 3^2$	
$\frac{1}{9} = 3^{-2}$	
$1 = e^0$	
$a = b^c$	

Example 1.26. The graph of the Logarithmic Function $y = \log_b x$ $(b > 1)$.
Because the logarithmic function and the exponential function are inverse of one another, we can obtain the graph of $y = \log_b x$ by reflecting the graph $y = b^x$ about the line $y = x$



Domain:...................... Range:............................

y-intercept:................. x-intercept:................... Asymptote: ....

1.27. Properties of Logarithms

(a) $\log_b b = 1$

(b) $\log_b 1 = 0$

(c) $\log_b(u \times v) = \log_b u + \log_b v$

(d) $\log_b \frac{u}{v} = \log_b u - \log_b v$

(e) $\log_b a^n = n \log_b a$

(f) $b^{\log_b x} = x$

(g) $\log_b b^x = x$

(h) $\log_a x = \frac{\log_b x}{\log_b a}$

1.3 New Functions from Old

Definition 1.28. (Arithmetical Operations on Functions): Two functions, f and g can be added, subtracted, multiplied, and divided in a natural way to form new functions $f+g$, $f-g$, fg, and f/g.

Let f and g be two functions. Then the sum $f+g$, the difference $f-g$, the product fg and the quotient f/g are function defined by the following definition.

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (f-g)(x) &= f(x)-g(x) \\ (fg)(x) &= f(x)\cdot g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \text{ provided that } g(x) \neq 0. \end{aligned}$$

The new domain is the intersection of the domains of f and g. Of course, for the function f/g we also exclude the points where $g(x) = 0$.

Example 1.29. Let $f(x) = \sqrt{x+2}$ and $g(x) = 3 - 2x$. Compute the the domain and formulas for the followings

(a) $(f+g)(x)$
Solution:

(b) $(f-g)(x)$
Solution:

(c) $(fg)(x)$
Solution:

(d) $\left(\frac{f}{g}\right)(x)$
Solution:

Definition 1.30. Composition of Function: Given function f and g, the composition of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those input x in the domain of g for which $g(x)$ is in the domain of f.

- In general, the functions $f \circ g$ and $g \circ f$ are *not* the same.
- For descriptive purposes, we will refer to g as the "inside function" and f as the "outside function" in the expression $f(g(x))$.
  - The inside function performs the first operation and the outside function performs the second.
- There is always more than one way to express a function as a composition.
- The composition of functions is always associative. That is, $f \circ (g \circ h) = (f \circ g) \circ h$. Since there is no distinction between the choices of placement of parentheses, they may be safely left off.
  - $(f \circ g \circ h)(x) = f(g(h(x)))$.

Example 1.31. Let $f(x) = x + 1$, $g(x) = \frac{1}{x}$, and $h(x) = x^2$ Compute the the domain (if possible) and formulas for the followings

(a) $(f \circ g)(x)$
Solution:

(b) $(g \circ f)(x)$
Solution:

(c) $(f \circ f)(x)$
Solution:

(d) $(f \circ g \circ h)(x)$
Solution:

1.32. Many problems in mathematics are attacked by "decomposing" functions into compositions of simpler functions.

Example 1.33. Express the function h as a composition of two simple functions f and g.

(a) $h(x) = \sqrt{1 - x^2}$
Solution:

(b) $h(x) = e^{x^3+1}$
Solution:

1.34. Geometric effects of performing basic operations on functions: Let c be a positive number.

Equation	Geometric Effect
$y = f(x) + c$	Translate the graph of $y = f(x)$ up c units
$y = f(x) - c$	Translate the graph of $y = f(x)$ down c units
$y = f(x + c)$	Translate the graph of $y = f(x)$ left c units
$y = f(x - c)$	Translate the graph of $y = f(x)$ right c units
$y = -f(x)$	Reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$	Reflect the graph of $y = f(x)$ about the y-axis
$y = cf(x)$, $c > 1$	Stretch the graph of $y = f(x)$ vertically by a factor of c
$y = cf(x)$, $0 < c < 1$	Compress the graph of $y = f(x)$ vertically by a factor of $1/c$
$y = f(cx)$, $c > 1$	Stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = f(cx)$, $0 < c < 1$	Compress the graph of $y = f(x)$ horizontally by a factor of $1/c$

Example 1.35. Sketch the graph of $y = (x - 2)^2 + 1$
Solution:

Example 1.36. Sketch the graph of $y = |x+2| - 1$
Solution:

Example 1.37. Sketch the graph of $y = \sqrt{-x}$
Solution:

1.38. We have two approaches to sketch functions of the form $y = f(mx+c)$

(a) $f(x) \xrightarrow{x \to mx} f(mx) \xrightarrow{x \to x + \frac{m}{c}} f\left(m\left(x + \frac{c}{m}\right)\right) = f(mx+c)$.

(b) $f(x) \xrightarrow{x \to x+c} f(x+c) \xrightarrow{x \to mx} f(mx+c)$.

Example 1.39. Sketch the graph of $y = \sqrt{2-x}$
Solution:

1.40. Symmetry (This idea can be applied to all curve in the xy-plane. In particular, it is not limited to functions.)

(a) Symmetry about the x-axis

- $\equiv$ For each point (x, y) on the graph, the point $(x, -y)$ is also on the graph.
- $\equiv$ Replacing all y by $-y$ in the graphs's equation produces an equivalent equation.

(b) Symmetry about the y-axis

- $\equiv$ For each point (x, y) on the graph, the point $(-x, y)$ is also on the graph.
- $\equiv$ Replacing all x by $-x$ in the graphs's equation produces an equivalent equation.

(c) Symmetry about the origin

- $\equiv$ For each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.
- $\equiv$ Rotating the graph $180\,\text{deg}$ about the origin leaves it unchanged.

1.41. Even and odd functions:

(a) Even function

$\equiv$ $f(-x) = f(x)$.

$\equiv$ The graph is symmetric about the y-axis.

(b) Odd function

$\equiv$ $f(-x) = -f(x)$.

$\equiv$ The graph is symmetric about the origin.

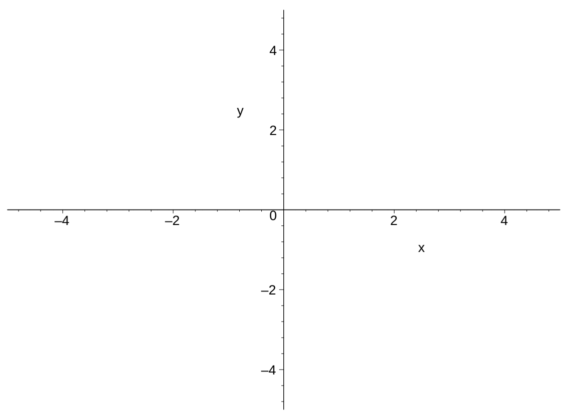
1.4 Family of Functions

1.42. Functions are often grouped into families according to the form of their defining formulas or other common characteristics.

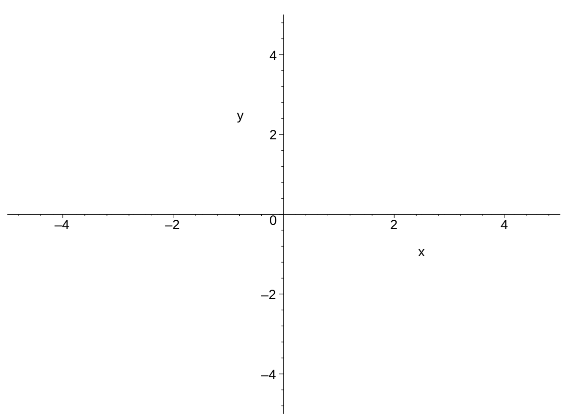
- Constants that are varied to produce families of curves are called **parameters**.

1.43. The family of lines

Recall that the equation $y = mx + b$ represents a line of slope m and y-intercept b. If we keep b fixed but change m, the graph of this family looks like:



However if we keep m fixed but change b, the graph of this family looks like:

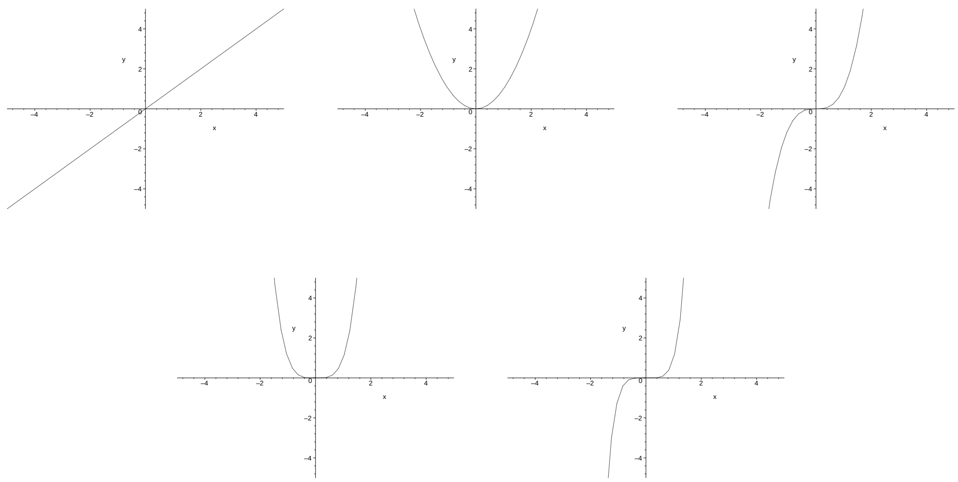


1.44. The family of x^n

Recall that a power function is a function of the form

$$f(x) = kx^r$$

where k and r are any real numbers.The followings are graphs of power functions with various degree:



1.45. The family of Polynomial function

Recall that A polynomial function of degree n is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \text{ with } a_n \neq 0$$

where $a_n, a_{n-1}, ..., a_1, a_0$ are constants and n is a non-negative integer.

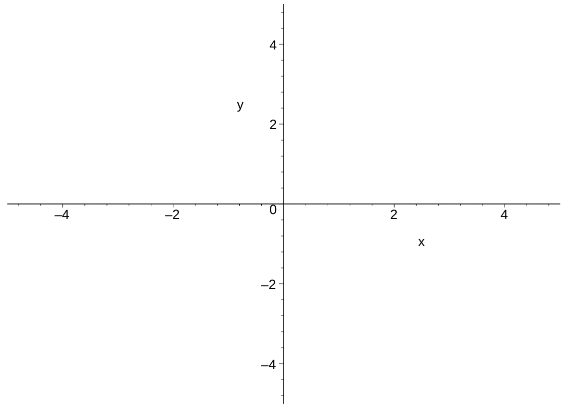
- Nonzero constant polynomials are considered to have degree 0.
- Polynomials of degree 1, 2, 3, 4, and 5 are described as linear, quadratic, cubic, quartic, and quintic, respectively.
- The natural domain of a polynomial is $(-\infty, +\infty)$.
- The number of peaks and valleys is less than the degree of the polynomial.

Example 1.46. Sketch the graph of $y = ax^2$

(a) The graph of $y = ax^2$ is a parabola with vertex at the origin. It is similar in shape to $y = x^2$.

(b) The parabola $y = ax^2$ opens upward if $a > 0$, downward if $a < 0$.

(c) The parabola $y = ax^2$ is narrower than $y = x^2$ if $|a| > 1$, wider than $y = x^2$ if $|a| < 1$.


4
y
2
0
–4
–2
2
4
x
–2
–4

2 Limits and Continuity

The concept of a "limit" is the fundamental building block on which all calculus concepts are based.

2.1. The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value.

Example 2.2. Consider the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

Notice that the point $x = 2$ is not in the domain therefore we can't find the value of $f(2)$. However we can find the "limit" of $f(x)$ at $x = 2$.

2.1 Definition of Limits

Definition 2.3 ("Informal" Definition of (Two-Sided) Limits)**.**

- $\equiv$ The values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a)
- $\equiv$ $\lim_{x \to a} f(x) = L$
- $\equiv$ The "limit of $f(x)$ is L as x approaches a from either side"
- $\equiv$ $f(x) \to L$ as $x \to a$.
- $\equiv$ "$f(x)$ approaches L as x approaches a."

2.4. Remarks:

- Since x is required to be different from a, the value of f at a, or even whether f is defined at a, has no bearing on the limit L. In other words, the limit describes the behavior of f **close to** a but not at a.
- The limit defined here is called a **two-sided limit** because it requires the values of $f(x)$ to get closer and closer to L as values of x are taken from *either side* of $x = a$.
- In general, there is no guarantee that a function f will have a (two-sided) limit at a given point a.

Example 2.5. Find $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$.
Solution:

Example 2.6. Guess $\lim_{x\to 0} \frac{\sin x}{x}$.
Solution:

Example 2.7. Guess $\lim_{x\to 0.1} \sin \frac{1}{x}$.
Solution:

2.8. If the values of $f(x)$ do not get closer and closer to *any single real number* L as $x \to a$, then we say that $\lim_{x\to a} f(x)$ **does not exist (DNE)**.

- ∞ is NOT a real number.

Example 2.9. Guess $\lim_{x\to 0} \sin \frac{1}{x}$.
Solution:

2.10. Some functions exhibit different behaviors on the two sides of an x-value a, in which case it is necessary to distinguish whether values of x near a are on the left side or on the right side of a for the purposes of investigating limiting behavior.

Example 2.11. Find $\lim_{x\to 0} \frac{|x|}{x}$.

Definition 2.12. One-sided Limits:

(a) Limit from the right

$\equiv$ The values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but *greater* than a)

$\equiv$ $\lim_{x\to a^+} f(x) = L$

$\equiv$ "The limit of $f(x)$ as x approaches a from the right is L."

$\equiv$ "$f(x)$ approaches L as x approaches a from the right."

$\equiv f(x) \to L$ as $x \to a^+$

(b) Limit from the left

$\equiv$ The values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but *less* than a)

$\equiv \lim_{x \to a^-} f(x) = L$

$\equiv$ "The limit of $f(x)$ as x approaches a from the left is L."

$\equiv$ "$f(x)$ approaches L as x approaches a from the left."

$\equiv f(x) \to L$ as $x \to a^-$

Example 2.13. $\lim_{x \to 0^-} \frac{|x|}{x} = -1$ and $\lim_{x \to 0^+} \frac{|x|}{x} = 1$.

2.14. The two-sided limit of a function $f(x)$ **exists** at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

- If the right-handed limit at $x = a$ and the left-handed limit at $x = a$ are not equal, then limit of $f(x)$ ***does not exist***.

Example 2.15. Let

$$f(x) = \begin{cases} 4 - x & \text{if } x < 2; \\ 1 & \text{if } x = 2 \\ x^2 - 2 & \text{if } x > 2 \end{cases}$$

Find

$f(2)$ = ... $\lim_{x \to 2^+} f(x)$ = ...

$\lim_{x \to 2^-} f(x)$ = ... $\lim_{x \to 2} f(x)$ = ...

Example 2.16. Let

$$f(x) = \begin{cases} x^2 + 4 & \text{if } x < 1 \\ 5 & \text{if } x = 1 \\ x^3 & \text{if } x > 1 \end{cases}$$

Find

$f(1)$ = ... $\lim_{x \to 1^+} f(x)$ = ...

$\lim_{x \to 1^-} f(x)$ = ... $\lim_{x \to 1} f(x)$ = ...

Definition 2.17. Infinite Limits

(a) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \text{ and } \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ **increases without bound** as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty.$$

- Equivalently, we say that $\lim_{x \to a} f(x) = +\infty$ if for any specified number M, the values of $f(x)$ will exceed M for positive values of x taken near enough to a.

(b) The expressions

$$\lim_{x \to a^-} f(x) = -\infty \text{ and } \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ **decreases without bound** as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty.$$

- The symbols $+\infty$ and $-\infty$ here are not real numbers; they simply describe particular ways in which the limits fail to exist.
  - Do not make the mistake of manipulating these symbols using rules of algebra.
    - For example, it is incorrect to write $(+\infty) - (+\infty) = 0$.

Example 2.18. $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Definition 2.19. When we have a one-sided infinite limit at a, the graph of $y = f(x)$ either rise or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit. The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$.

Example 2.20. Is it true that when $\lim_{x \to a^-} f(x) = +\infty$, the limit on another side $\lim_{x \to a^-} f(x)$ must be either $+\infty$ or $-\infty$?
Solution:

2.2 Computing Limit

2.21. Some basic limits: Let c and k be real numbers

$$\lim_{x \to c} k = k$$
$$\lim_{x \to c} x = c$$
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Theorem 2.22. Rules for Limits
Assume that

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M.$$

(a) The limit of a sum is the sum of the limits:

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M.$$

(b) The limit of a difference is the difference of the limits:

$$\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M.$$

(c) A constant factor can be moved through a limit symbol:

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x).$$

(d) The limit of a product is the product of the limits:

$$\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M.$$

(e) The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \quad M \neq 0.$$

(f) The limit of an nth root is the nth root of the limit:

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}, \text{ provided } L > 0 \text{ if } n \text{ is even.}$$

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

Theorem 2.23. The limit of a Polynomial Function
If $p(x)$ is any polynomial and c is any number, then

$$\lim_{x\to c} p(x) = p(c)$$

Example 2.24. Find:

$$\lim_{x\to 1} x^2 - 2x + 3 = \text{...}$$
$$\lim_{x\to 0} x^2 - 2x + 3 = \text{...}$$
$$\lim_{x\to -1} x^2 - 2x + 3 = \text{...}$$

2.25. Let

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(a) If $q(a) \neq 0$, then $\lim_{x \to a} f(x) = f(a)$.

(b) If $q(a) = 0$ but $p(a) \neq 0$, then $\lim_{x \to a} f(x)$ does not exist and one of the following situations occur:

   (i) The limit may be $-\infty$, e.g. $\lim_{x \to a} \frac{1}{(x-a)^2} = +\infty$.

   (ii) The limit may be $+\infty$, e.g. $\lim_{x \to a} -\frac{1}{(x-a)^2} = -\infty$.

   (iii) The limit may be $-\infty$ from one side and $+\infty$ from the other, e.g. $\lim_{x \to a^+} \frac{1}{x-a} = +\infty$ and $\lim_{x \to a^-} \frac{1}{x-a} = -\infty$.

(c) If $q(a) = p(a) = 0$, the numerator and denominator must have one or more common factors of $x - a$. In this case, the limit can be found be cancelling all common factors of $x - a$.

Example 2.26. Evaluate the following limits

(a) $\lim_{x \to 1} \frac{x-1}{x+2}$
Solution:

(b) $\lim_{x \to 1} \frac{x+3}{x^2+2x-3}$
Solution:

(c) $\lim_{x \to 1} \frac{x^2-1}{x-1}$
Solution:

2.27. A quotient $\frac{f(x)}{g(x)}$ in which $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ is called an indeterminate form of type $0/0$. Sometimes, limits can be found by algebraic simplification. Frequently, this will not work and other methods must be used. For example, consider $\lim_{x \to 0} \frac{\sin x}{x}$.

Example 2.28. Evaluate the following limits:

(a) $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$
Solution:

(b) $\lim_{x \to 1} \frac{|x-1|}{x-1}$
Solution:

2.3 Limits at Infinity

In this section we will be concerned with the behavior of $f(x)$ as x increases or decreases without bound.

Definition 2.29. If the values of x increase without bound, then we write $x \to +\infty$, and if the values of x decrease without bound, then we write $x \to -\infty$. The behavior of a function $f(x)$ as x increases without bound or decreases without bound is sometimes called the **end behavior** of the function.

Definition 2.30. An informal view of limits at infinity:

(a) If the values of $f(x)$ eventually get as close as we like to a number L as x increases without bound, then we say that L is the limit of $f(x)$ as x approaches $+\infty$, and we write

$$\lim_{x \to \infty} f(x) = L$$

(b) If the values of $f(x)$ eventually get as close as we like to a number L as x decreases without bound, then we say that L is the limit of $f(x)$ as x approaches $-\infty$, and we write

$$\lim_{x \to -\infty} f(x) = L$$

- In either case, we call the line $y = L$ a **horizontal asymptote** for the graph of f.

Example 2.31. $\lim_{x \to +\infty} \frac{1}{x} = 0$ and $\lim_{x \to -\infty} \frac{1}{x} = 0$.

2.32. The limit laws in Theorem 2.22 carry over without change to limits at $+\infty$ and $-\infty$. Here are some additional properties.

(a) $\lim_{x \to +\infty} x^n = +\infty$, $n = 1, 2, 3, \ldots$

(b) $\lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ \infty, & n = 2, 4, 6, \ldots \end{cases}$

(c) The end behavior of a polynomial matches the end behavior of its highest degree term.

(d) $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$ for $n > 0$.

Example 2.33. Find $\lim_{x \to -\infty} \sqrt{9 + \frac{1}{x^2}}$

Solution:

2.34. To find the limit at infinity of a rational function, divide the numerator and denominator by the highest power of x that occurs in the denominator. This is helpful because the powers of x will be transformed to powers of $1/x$.

Example 2.35. Evaluate the following limits

(a) $\lim_{x \to -\infty} \frac{x^4 - 3x^2 + 1}{3x^2 - 4x^4}$
Solution:

(b) $\lim_{x \to \infty} \frac{3x}{x^2 + 6}$
Solution:

(c) $\lim_{x \to \infty} \frac{3x^2}{x^2 + 6}$
Solution:

We conclude that the end behavior of a rational function matches the end behavior of the quotient of the highest degree term in the numerator divided y the highest degree term in the denominator.

2.36. The idea of transforming powers of x into powers of $1/x$ can be applied to other kinds of quotient.

Example 2.37. Evaluate the following limits:

(a) $\lim_{x \to \infty} \frac{2x+2}{\sqrt{3x^2 - 1}}$
Solution:

(b) $\lim_{x \to \infty} \frac{x+1}{\sqrt{x+2}}$
Solution:

Example 2.38. Evaluate the following limits:

(a) $\lim_{x \to \infty} \sqrt{x^2 + 5} - x$
Solution:

(b) $\lim_{x \to \infty} \sqrt{x^2 + 5x} - x$
Solution:

Example 2.39. Some more interesting limits:

(a) $\lim_{x\to\infty} \sqrt{x^2 + a_1 x + a_0} - x = \frac{a_1}{2}$

(b) $\lim_{x\to\infty} \left(\sqrt{x^2 + a_1 x + a_0} - \sqrt{x^2 + b_1 x + b_0}\right) = \frac{a_1}{2} - \frac{b_1}{2}$

(c) $\lim_{x\to\infty} \left(\sqrt{x^4 + 2b_1 x^3 + a_2 x^2 + a_1 x + a_0} - \left(x^2 + b_1 x + b_0\right)\right) = \frac{a_2}{2} - \frac{b_1^2}{2} - b_0$

Example 2.40. Find (if exists) horizontal asymptotes of functions in Example 2.35.

2.41. The limits at infinity for a nonzero periodic function do not exist.

Example 2.42. For the sin and cos function, the limits at infinity fail to exist not because $f(x)$ increases or decreases without bound, but rather because the values vary between -1 and 1 without approaching some specific real umber.

Definition 2.43 (Formal Definition of Limit)**.** Let $f(x)$ be defined for all x in some open interval containing the number a, the the possible exception that $f(x)$ need not be defined at a. We will write

$$\lim_{x\to a} = L$$

if given any number $\varepsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ if } 0 < |x - a| < \delta.$$

- Attributed to K. Weierstrass.
- Commonly called the "epsilon-delta" definition of a two-sided limit.

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MAS 116: Lecture Notes 2

Semester: 3/2008
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2.44. The Symbol $\pm\infty$: Using "$\pm\infty$" in a sentence gives us two statements. For example,

$$\lim_{x\to\pm\infty} f(x) = L$$

says that

$$\lim_{x\to+\infty} f(x) = L \text{ and } \lim_{x\to-\infty} f(x) = L.$$

2.45. Polynomial End Behavior: If

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad (a_n \neq 0),$$

then $y = a_n x^n$ is an end behavior model of f.

- $\lim_{x\to\pm\infty} \frac{f(x)}{a_n x^n} = 1 \quad (a_n \neq 0)$.

2.46. Rational Function End Behavior: Consider a rational function $f(x) = \frac{p(x)}{h(x)}$,

(a) If $\deg p < \deg h$, then $\lim_{x\to\pm\infty} f(x) = 0$ and the line $y = 0$ is the horizontal asymptote.

(b) If $\deg p = \deg h$, then $\lim_{x\to\pm\infty} f(x) = k$ for some constant k and the line $y = k$ is the horizontal asymptote.

- Suppose a and b are the leading coefficients of $p(x)$ and $h(x)$, respectively. Then, $k = a/b$.

(c) If $\deg p > \deg h$, then $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ can be $-\infty$ or $+\infty$.

- Suppose $q(x)$ and $r(x)$ are the quotient and remainder when $p(x)$ is divided by $h(x)$; that is,

$$f(x) = q(x) = \frac{r(x)}{h(x)} \text{ with } \deg r < \deg h.$$

The graph of q is called the **end behavior asymptote** of f.

2.47. The Squeezing Theorem: Let f, g, and h be functions satisfying

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then f also has this limit as x approaches c, that is,

$$\lim_{x \to c} f(x) = L.$$

- Remark: Same idea works for the case of limit at infinity.

Example 2.48. Evaluate the following limits:

(a) $\lim_{x \to 0} x \sin \frac{1}{x}$
Solution:

(b) $\lim_{x \to \infty} \frac{\sin(x)}{x}$
Solution:

2.4 Continuity

2.49. Intuitively, the graph of a function can be described as a continuous functions if its graph has no breaks or holes. In particular, a function $y = f(x)$ whose graph can be sketched over any interval of its domain with one continuous motion of the pencil is an example of a **continuous function**.

Definition 2.50 (Continuity at an Interior Point)**.** A function f is continuous at $x = c$ if all three of the following statements are true:

(1) $f(c)$ is defined.

(2) $\lim_{x \to c} f(x)$ exists.

(3) $\lim_{x \to c} f(x) = f(c)$

- If one or more of the conditions of this definition fails to hold, then we will say that f has a **discontinuity at** $x = c$ or that f is discontinuous at c. In this case, we call c a point of discontinuity of f. Note that c may or may not be in the domain of f.

- The third condition in Definition 2.50 implicitly implies the first two.

Example 2.51. Referring to Figure 2.5.1 in the textbook, we see that the graph of a function has a break or hole if any of the following conditions occur:

- The function f is undefined at c (Figure 2.5.1a).
- The limit of $f(x)$ does not exist as x approaches c (Figure 2.5.1b, 2.5.1c).
- The value of the function and the value of the limit at c are different (Figure 2.5.1d).

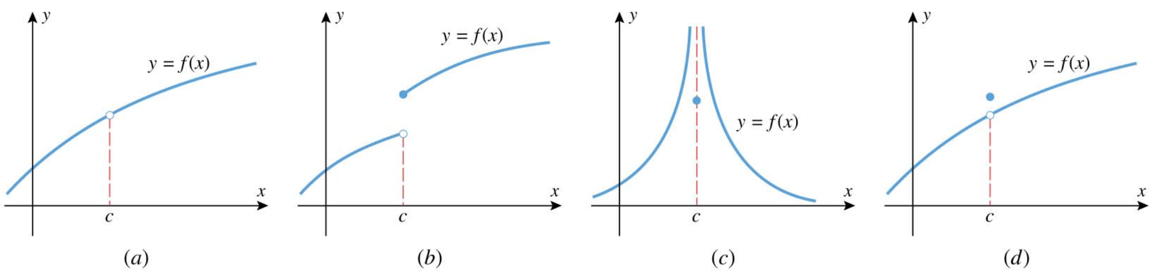


Figure 2.5.1

Example 2.52. Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2-4}{x-2}, \quad g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}, \text{ and } h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

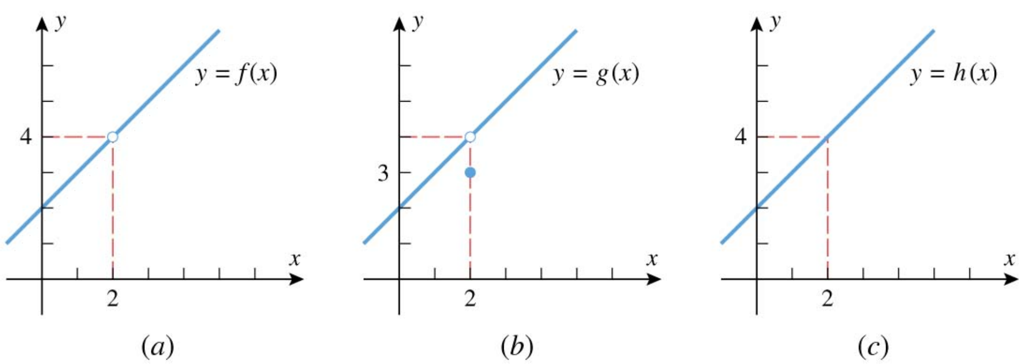


Figure 2.5.2

Solution: In each case, we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases, the functions are identical except at $x = 2$, and hence all three have the same limit at $x = 2$.

Definition 2.53 (Continuity at an Endpoint)**.** A function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that endpoint. In particular, a function $y = f(x)$ is continuous at a left endpoint a or a right endpoint b of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b),$$

respectively.

Definition 2.54. Continuity on an interval:

(a) A function f is said to be **continuous on an open interval** (a, b) if it is continuous at each number in an open interval (a, b).

- This definition applies to infinite open interval of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$.
- In the case where f is continuous on $(-\infty, +\infty)$. we say that f is **continuous everywhere**.

(b) A function f is said to be continuous on a closed interval $[a, b]$ if the following conditions are satisfied:

(i) f is continuous on (a, b)

(ii) f is continuous from the left at b i.e.

$$\lim_{x\to b^-} f(x) = f(b)$$

(iii) f is continuous from the right at a i.e.

$$\lim_{x\to a^+} f(x) = f(a)$$

Example 2.55. What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$
Solution:

Example 2.56. Find the value k such that the following function is continuous on any interval.

$$f(x) = \begin{cases} \frac{5x^2-10x}{x-2} & \text{if } x > 2; \\ kx & \text{if } x \leq 2 \end{cases}$$

Solution:

2.5 Some Properties of Continuous Functions

2.57. The following function are continuous wherever they are defined.

(a) Polynomial functions

(b) Rational functions

(c) Power function

(d) Trigonometric functions

(e) Exponential functions of the form $a^x, a > 0$ and $a \neq 1$

(f) Logarithm function of the form $\log_a x, a > 0$ and $a \neq 1$

The phrase *wherever they are defined* is crucial. It help us identify points where a function might be discontinuous. For example, a rational function is continuous at every point where the denominator is nonzero and has discontinuities at the points where the denominator is zero.

2.58. Using the limit laws, we find that the following hold for combination of continuous functions.

Suppose that a is a constant and the functions f and g are continuous at $x = c$. Then the following functions are continuous at $x = c$

(a) $a \cdot f$

(b) $f + g$

(c) $f - g$

(d) $f \cdot g$

(e) $\frac{f}{g}$, provided $g(c) \neq 0$

Example 2.59. Find values of x, if any, at which f is not continuous.

(a) $f(x) = 2x^3 - 3x + 1$
Solution:

(b) $f(x) = \frac{\pi}{x}$
Solution:

(c) $f(x) = \frac{x^2+x+1}{x-2}$
Solution:

(d) $f(x) = \sqrt{x}$
Solution:

Example 2.60. The function $f(x) = |x|$ is continuous everywhere.
Solution:

2.61. Limits and continuity involving composition of functions

(a) A limit symbol can be mode through a function sign provided that the limit of the expression inside the function sign exists and the function is continuous at this limit. More specifically, if $\lim_{x\to c} g(x) = L$ and if the function f is continuous at L, then $\lim_{x\to c} f(g(x)) = f(L)$. That is

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right).$$

- We do not require $g(x)$ to be continuous at $x = c$; only its limit at $x = c$ needs to exist.
- This equality remains valid if $\lim_{x\to c}$ is replaced everywhere by one of $\lim_{x\to c+}$, $\lim_{x\to c-}$, $\lim_{x\to +\infty}$, or $\lim_{x\to -\infty}$.
- $\lim_{x\to c} |g(x)| = \left|\lim_{x\to c} g(x)\right|$ provided $\lim_{x\to c} g(x)$ exists.

(b) If a function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

(c) If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

Example 2.62. Find

$$\lim_{x \to 0} e^{x-1}$$

Solution:

Example 2.63. Find values of x, if any, at which f is not continuous.

(a) $f(x) = \sin\left(\frac{\pi}{x}\right)$
Solution:

(b) $f(x) = \ln(x + 1)$
Solution:

Example 2.64. Can the absolute value of a function that is not continuous everywhere be continuous everywhere?
Solution:

3 The Derivative

3.1. If the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists, then it can be interpreted either

(a) as the slope of the tangent line to the curve $y = f(x)$ at $x = x_0$ or

(b) as the instantaneous rate of change of y with respect to x at $x = x_0$.

This limit is so important that it has a special notation:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

You can think of f' (read "f prime") as a function whose input is x_0 and whose output is the number $f'(x_0)$ that represents either the slope of the tangent line to $y = f(x)$ at $x = x_0$ or the instantaneous rate of change of y with respect to x at $x = x_0$. To emphasize this function point of view, we will replace x_0 in (1) by x and make the following definition.

Definition 3.2. The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{2}$$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit exists.

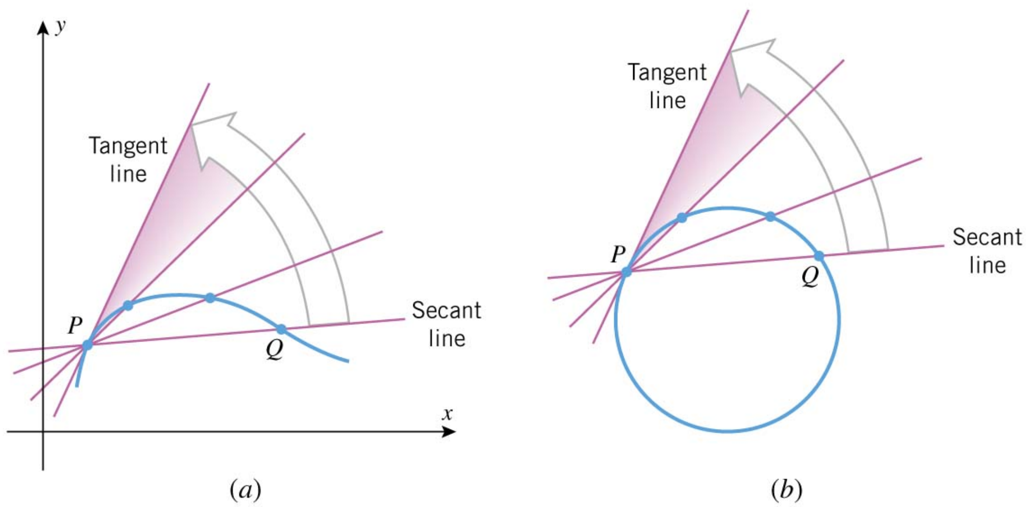
- The term "derivative" is used because the function f' is derived from the function f by a limiting process.
- The expression

  $$\frac{f(x + h) - f(x)}{h}$$

  that appears in (2) is commonly called the "difference quotient".

3.1 Tangent Lines and Rates of Change

3.3. Consider a point P on a curve in the xy-plane. Suppose that Q is any point that lies on the curve and is different from P. The line through P and Q is called a **secant line** for the curve at P. Intuition suggests that if we move the point Q along the curve toward P, then the secant line will rotate toward a limiting position. The line in this limiting position is what we will consider to be the **tangent line** at P.



(a) (b)

Figure 2.1.4

3.4. The **secant line** for the curve $y = f(x)$ at the point $P(x_0, f(x_0)))$ which passes through the point $Q(x_1, f(x_1))$ is the line with equation

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

Definition 3.5. Suppose that x_0 is in the domain of the function f. The **tangent line** to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

$$y - f(x_0) = m_{\tan}(x - x_0)$$

where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{3}$$

provided the limit exists.

- For simplicity, we will also call this the tangent line to $y = f(x)$ at x_0.
- Alternative way of expressing Formula (3) by letting $h = x - x_0$:

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) \tag{4}$$

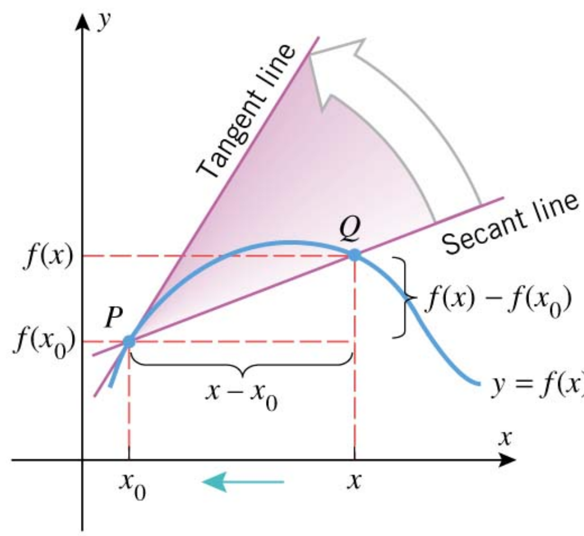


Figure 3.1.1

Example 3.6. Find the derivative with respect to x of $f(x) = x^2 + 1$, and use it to find the equation of the tangent line to $y = x^2 + 1$ at $x = 2$.
Solution:

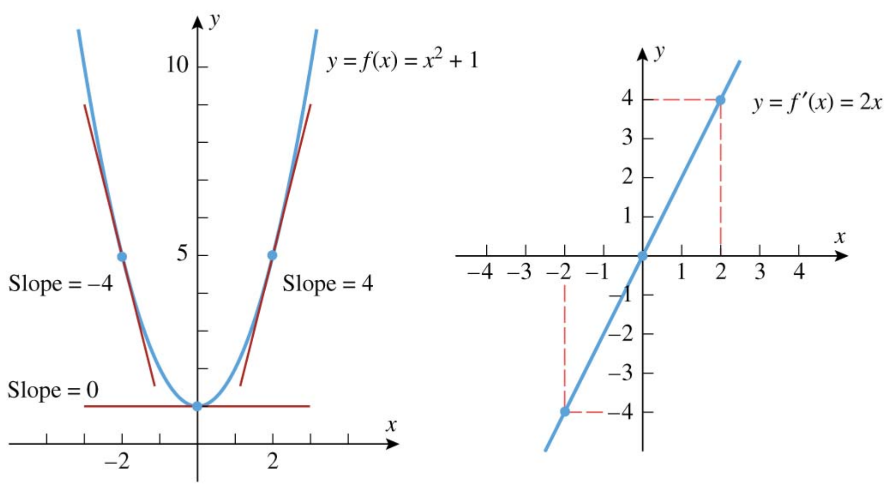


Figure 3.2.2

Example 3.7. Find an equation for the tangent line to the curve $y = 3/x$ at the point $(2, 1)$ on this curve.

Solution:

Example 3.8. Given that $f(-2) = 3$ and $f'(-2) = -4$, find an equation for the tangent line to the graph of $y = f(x)$ at $x = -2$
Solution:

Definition 3.9. Rate of Change: Suppose $y = f(x)$.

(a) We define the **average rate of change** r_{ave} of y with respect to x over the interval $[x_0, x_1]$ to be the quotient

$$\frac{\text{change in y}}{\text{change in x}} = \frac{\Delta x}{\Delta y} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

- Geometrically, it is the slope of the secant line from the point $(x_0, f(x_0))$ to the point $(x_1, f(x_1))$.

(b) The **instantaneous rate of change** of y with respect to x at $x = x_0$ is given by

$$r_{\text{inst}} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{5}$$

if this limit exists.

- Geometrically, it is the slope of the tangent line at the point $(x_0, f(x_0))$.
- If desired, we can let $h = x - x_0$, and rewrite (5) as

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0). \tag{6}$$

3.2 Differentiability

Derivative is the primary mathematical tool that is used to calculate and study rates of change.

Definition 3.10. A function f is said to be differentiable at x_0 if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

exists. If f is **differentiable** at each point of the open interval (a, b), then we say that it is differentiable on (a, b), and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is **differentiable everywhere**.

- The process of finding a derivative is called differentiation.
  - You can think of differentiation as an operation on functions that associates a function f' with a function f.
- When the independent variable is x, the differentiation operation is also commonly denoted by
$$f'(x) = \frac{d}{dx}[f(x)] \quad \text{or} \quad f'(x) = D_x[f(x)].$$
- In the case where there is a dependent variable $y = f(x)$, the derivative is also commonly denoted by
$$f'(x) = y'(x) \quad \text{or} \quad f'(x) = \frac{dy}{dx}.$$
- With the above notations, the values of the derivative at a point x_0 can be expressed as

$$f'(x_0), \quad \left.\frac{d}{dx}[f(x)]\right|_{x=x_0}, \quad D_x[f(x)]|_{x=x_0}, \quad y'(x_0), \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=x_0}.$$

Theorem 3.11. If $y = f(x)$ has a derivative at $x = c$, then $f(x)$ is continuous at $x = c$.

- Contrapositive form: If a function f is not continuous at x_0, then f is not differentiable at x_0.
- A function may be continuous at a point but not differentiable at that point.
  - This occurs, for example, at corner points of continuous functions.
    - $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there.

Example 3.12. Find the derivative of $f(x) = x^2$ with respect to x
Solution:

Example 3.13. Find the derivative of $f(x) = 1/x$ with respect to x
Solution:

Example 3.14. Find the derivative of $f(x) = \sqrt{x}$ with respect to x
Solution:

Example 3.15. Let $f(x) = |x|$, find $f'(0)$, if exists.
Solution:

Example 3.16. The derivative fails to exist in the following circumstances.

(a) The graph of the function has a corner.

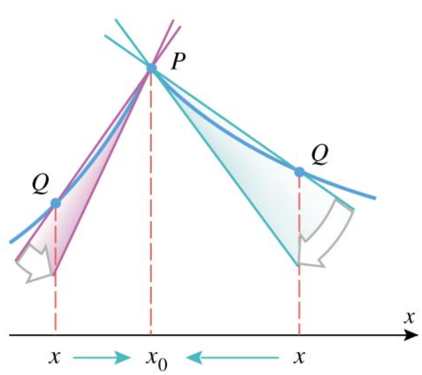


Figure 3.2.7

(b) The graph of the function has a vertical tangent.

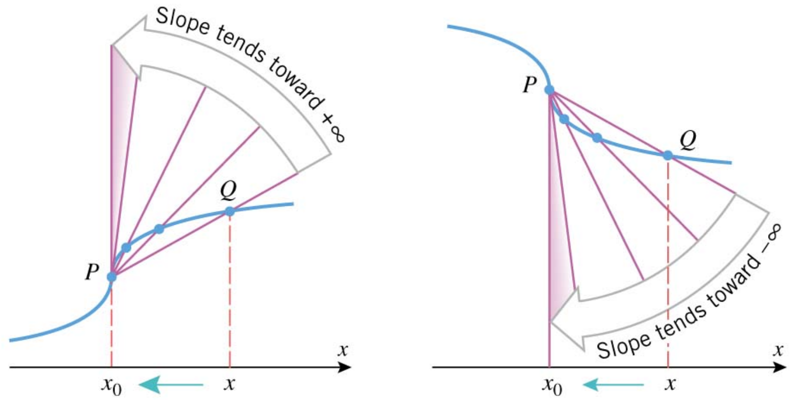


Figure 3.2.8

(c) The graph of the function has a jump (discontinuity.)

Example 3.17. There are other less obvious circumstances under which a function may fail to be differentiable. For example, the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$.