Sirindhorn International Institute of Technology
Thammasat University
Department of Common and Graduate Studies

MAS 116: Lecture Notes 9

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Instructors: Dr. Prapun Suksompong

10.3 Partial Sums and Convergence of Series

Definition 10.16. An **infinite series** is an expression that can be written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- The numbers $a_1, a_2, a_3, \ldots$ are called the **terms** of the series.

Let s_n denote the sum of the first n terms of a series. Thus,

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n.$$

The number s_n is called the nth partial sum of the series.

- The sequence $\{s_n\}_{n=1}^{+\infty}$ is called the **sequence of partial sums**.
- $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$.
- In everyday language, the words "sequence" and "series" are often used interchangeably. However, in mathematics there is a difference between the two terms–a sequence is a succession where as a series is a sum.

Example 10.17. The most familiar examples of series occur in the decimal representations of real numbers. For example, when we write $\frac{1}{3}$ in the decimal form $\frac{1}{3} = 0.3333\ldots$, we mean

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

which suggest that the decimal representations of $\frac{1}{3}$ can be viewed as a sum of infinitely many real numbers.

Definition 10.18. Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$. If the sequence $\{s_n\}$ **converges** to a limit S, then the series is said to converge to S, and S is called the **sum** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} a_k.$$

If the sequence of partial sum diverges, then the series is said to **diverge**. A divergent series has no sum.

Example 10.19. Telescoping Sum: Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges or diverges. If it converges, find the sum.
Solution: We will first try to rewrite the partial sum in closed form:

$$\sum_{k=1}^{n} \frac{1}{k\,(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = 1 - \frac{1}{n+1}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{k\,(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k\,(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Definition 10.20. A **geometric series** is one in which each term is obtained by multiplying the preceding term by some fixed constant. If the initial term of the series is a and each term is obtained by multiplying the preceding term by r, then the series has the form

$$a + ar + ar^2 + ar^3 + ... + ar^n + \cdots$$

The number r is called the ratio for the series.

Theorem 10.21. A geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + ... + ar^n + \cdots \quad (\text{a} \neq 0)(\text{a} \neq 0)(\text{a} \neq 0)(\text{a} \neq 0)$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

- When $r \neq 1$, the nth partial sum is given by $s_n = a\frac{1-r^{n+1}}{1-r}$. (This is easily derived from the fact that $s_n - rs_n = a - ar^{n+1}$.)

Example 10.22. Determine whether the following series is a geometric series. If so, determine whether the series converges.

(a) $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

(b) $3 - \frac{3}{2} + \frac{3}{3} - \frac{3}{4} + \frac{3}{5} - \frac{3}{6} + ...$

(c) $\sum\limits_{k=1}^{\infty} (-1)^{k+1} 2^k 3^{2-k} = \sum\limits_{k=1}^{\infty} (-1)(-1)^k 2^k \frac{9}{3^k} = (-9) \sum\limits_{k=1}^{\infty} \left(-\frac{2}{3}\right)^k = \frac{-9\left(-\frac{2}{3}\right)}{1-\left(-\frac{2}{3}\right)} = \frac{18}{5}$

10.23. Algebraic Properties of Infinite Series:

(a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(a_n + b_n)$. In which case,

$$\sum_{n=1}^{\infty}(a_n + b_n) \text{ converges to } \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

and

$$\sum_{n=1}^{\infty}(a_n - b_n) \text{ converges to } \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n.$$

(b) If k is a nonzero constant, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} ca_n$ both converges or both diverge. In the case of convergence,

$$\sum_{n=1}^{\infty} ka_n \text{ converges to } k\sum_{n=1}^{\infty} a_n.$$

(c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=K}^{\infty} a_n$ both converge or both diverge.

- From this fact, when stating general results about convergence or divergence of series, it is convenient to use the notation $\sum a_k$ as a generic template for a series, thus avoiding the issue of whether the sum begins with $k = 0$ or $k = 1$ or some other values.
- Although convergence is not affected when finitely many terms are deleted from the beginning of a convergent series, the *sum* of the series will usually changed by the removal of those term.

10.4 Convergence Tests

There are many convergence test. The skill of selecting a good test is developed through lots of practice. In some instances a test may be inconclusive, so another test must be tried.

Theorem 10.24. If the series $\sum a_k$ converge, then $\lim\limits_{k\to\infty} a_k = 0$.

Theorem 10.25. The Divergence Test:

$$\lim_{n\to\infty} a_n \neq 0 \text{ or does not exist then } \sum_{n=1}^{\infty} a_n \text{ diverges}$$

- If $\lim\limits_{n\to\infty} a_n = 0$, then the test is inconclusive.

Example 10.26.

(a) The series $\sum(1 - e^{-n})$ diverges because $\lim\limits_{n\to\infty}(1 - e^{-n}) = 1 \neq 0$.

(b) The series $\sum\limits_{k=1}^{\infty} \frac{k}{k+1}$ diverges because $\lim\limits_{k\to\infty} \frac{k}{k+1} = 1$.

Theorem 10.27. The (Improper) Integral Test: Let $\sum a_k$ be a series with positive terms. If f is a function that is decreasing and continuous on an interval $[a, +\infty]$ and such that $a_k = f(k)$ for all $k \geq a$, then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_a^{\infty} f(x)dx$$

both converge or both diverge.

- Caution: DO NOT erroneously conclude that the sum of the series is the same as the value of the corresponding integral.

Definition 10.28. The **harmonic series** is the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Example 10.29. Use the integral test to determine whether the following series converge or diverge.

(a) $\sum_{k=1}^{\infty} \frac{1}{k}$

(b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$

(c) $\sum_{n=0}^{\infty} \frac{3}{n+2}$

10.30. Notice that the harmonic series diverges, even though $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. The growth of its partial sum is slow, but they do in fact grow without bound.

Theorem 10.31. Convergence of p-Series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Example 10.32. Show that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{n^2} \right)$$

converges.

Theorem 10.33. Comparison Test: Suppose $0 \le a_n \le b_n$ for all n

(a) If $\sum b_n$ converges, then $\sum a_n$ converges.

(b) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

- It is not essential that the condition $a_n \le b_n$ hold for all k, as stated; the conclusions of the theorem remain true if this condition is eventually true.

Theorem 10.34. Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } 0 < c < \infty$$

then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Theorem 10.35. The Ratio Test: For a series $\sum a_n$, suppose the sequence of ratios $|a_{n+1}|/|a_n|$ has a limit:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

- If $L < 1$, then $\sum a_n$ converges.
- If $L > 1$, or if L is infinite, then $\sum a_n$ diverges.
- If $L = 1$, then this test is inconclusive.

Example 10.36. Show that the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

Theorem 10.37. The Root Test: For a series $\sum a_n$, suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n} = L$$

- If $L < 1$, then $\sum a_n$ converges.
- If $L > 1$, or if L is infinite, then $\sum a_n$ diverges.
- If $L = 1$, then the test is inconclusive.

10.5 Alternating Series

Definition 10.38. A series is called an **alternating series** if the terms alternate in sign.

Theorem 10.39. Alternating Series Test: A series of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 + \cdots$$

or

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converge if

$$0 < a_{n+1} \le a_n \text{ for all } n \text{ and } \lim_{n\to\infty} a_n = 0$$

- The partial sums oscillate back and forth, and since the distance between them tends to 0, they eventually converge.

Example 10.40. Show that the **alternating harmonic series** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

Definition 10.41. We say that the series $\sum a_n$ is

- **absolutely convergent** if $\sum |a_n|$ converges (which also means that $\sum a_n$ converges also).
- **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem 10.42. The Ratio Test for Absolute Convergence: For a series $\sum a_n$ with nonzero terms, suppose the sequence of ratios $|a_{n+1}|/|a_n|$ has a limit:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

- If $L < 1$, then $\sum a_n$ converges absolutely and therefore converges.
- If $L > 1$, or if L is infinite, then $\sum a_n$ diverges.
- If $L = 1$, then this test is inconclusive.

10.6 Maclaurin and Taylor Series

It is not a big step to extend the notions of Maclaurin and Taylor polynomials to series by not stopping the summation index at n.

Definition 10.43. If f has derivatives of all orders at $x = c$, then we called the series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

the **Taylor series for f about** $x = c$. In the special case where $c = 0$, this series becomes

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

in which case we call it the **Maclaurin seires for f**.

- The nth Maclaurin and Taylor polynomials are the nth partial sums for the corresponding Maclaurin and Taylor series.
- These series do not necessarily converge to $f(x)$ for all values of x.

10.44. It turns out that for $\cos x$, $\sin x$, and e^x, the Taylor series converge to the corresponding functions for all value of x, so we can write the following:

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + ... \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + ... + \frac{(-1)^n x^{2n}}{(2n)!} + ... \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ... + \frac{x^n}{n!} + ...
\end{aligned}$$

Example 10.45. By recognizing each series in problems below as a Taylor series evaluated at a particular value of x , find the sum of each of the following convergent series.

(a) $1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + ... + \frac{2^n}{n!} + \cdot$

(b) $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + ... + \frac{(-1)^n}{(2n+1)!} + \cdot$

10.7 Power Series and Interval of Convergence

Definition 10.46. If $c_0, c_1, c_2, \ldots$ are constants and x is a variable, then a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots \tag{30}$$

is called a **power series in** x.

- Every Maclaurin series is a power series in x.

Definition 10.47. If a numerical value is substituted for x in a power series, then the resulting series of numbers may either converge or diverge. The set of x-values for which a given power series in x converges is called the **convergence set**.

Theorem 10.48. The convergence set for a power series in x is always an interval centered at $x = 0$.

- For this reason, the convergence set of a power series in x is called the **interval of convergence**.
- In the case where the convergence set extends between $-R$ and R, we say that the series has **radius of convergence** R.

Theorem 10.49. For any power series in x, exactly one of the following is true:

(a) The series converges only for $x = 0$; the radius of convergence is defined to be $R = 0$.

(b) The series converges absolutely (and hence converges) for all values of x; the radius of convergence is defined to be $R = \infty$.

(c) The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$ and diverge if $x < -R$ or $x > R$; the radius of convergence has the value R. At either of the endpoints $x = R$ or $x = -R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

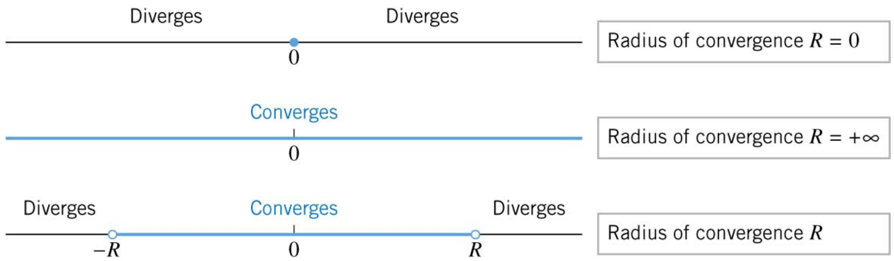

Figure 10.8.1

Example 10.50. Suppose that the power series $\sum c_n x^n$ converges when $x = -4$ and diverges when $x = 7$. Which of the following are true, false or not possible to determine?

(a) The power series converges when $x = 10$.

(b) The power series converges when $x = 3$.

(c) The power series converges when $x = 6$.

Example 10.51. Find the radius of convergence of the following power series:

(a) $\sum_{k=0}^{\infty} x^k$

(b) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

Definition 10.52. If a is a constant, and if x in (30) is replaced by $x - a$, the the resulting series has the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

- Any Taylor series about $x = a$ is a power series in $x - a$.

Theorem 10.53. For any power series $\sum c_k (x-a)^k$, exactly one of the following is true:

(a) The series converges only for $x = a$; the radius of convergence is defined to be $R = 0$.

(b) The series converges absolutely (and hence converges) for all values of x; the radius of convergence is defined to be $R = \infty$.

(c) The series converges absolutely (and hence converges) for all x in some finite open interval $(a - R, a + R)$ and diverge if $x < a - R$ or $x > a + R$; the radius of convergence has the value R. At either of the endpoints $x = a + R$ or $x = a - R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

Example 10.54. Suppose that the power series $\sum c_n(x-3)^n$ converges when $x = 7$ and diverges when $x = 9$. Which of the following are true, false or not possible to determine?

(a) The power series converges when $x = 11$.

(b) The power series converges when $x = 5$.

(c) The power series converges when $x = -3$.

Theorem 10.55. Method for Computing Radius of Convergence: To calculate the radius of convergence, R, for the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, use the ratio test with $a_n = (-1)^n c_n (x-a)^n$.

- If $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ is infinite then $R = 0$
- If $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$ then $R = \infty$
- If $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = K|x-a|$ where K is finite non zero then $R = \dfrac{1}{K}$

Note that the ratio test does not tell us anything if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ fails to exist, which can occur, for example, if some of the c_n are zero.

Definition 10.56. If a function f is expressed as a power series on some interval, then we say that the power series represents f on that interval.

10.57. Sometimes new functions actually originates as power series, and the properties of the functions are developed by working with their power series representations. For example, the functions

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \quad \text{and} \quad J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}(k!)(k+1)!},$$

which converge for all x, are called **Bessel functions**.