Sirindhorn International Institute of Technology
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MAS 116: Lecture Notes 8

Semester: 3/2008
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9 Taylor Polynomials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with.

9.1 Local Linear Approximations

9.1. Derivatives can be used to approximate nonlinear functions by linear functions. If a function f is differentiable at x_0, then a sufficiently magnified portion of the graph of f centered at the point $P(x_0, f(x_0))$ takes on the appearance of a straight line segment. Figure 3.9.1 illustrates this at several points on the graph of $y = x^2 + 1$.

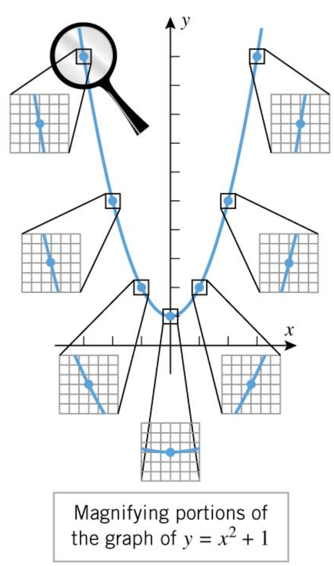

Figure 3.9.1

For this reason, a function that is differentiable at $x = c$ is sometimes said to be **locally linear** at c: locally, every differentiable curve behaves like a straight line.

9.2. As you can see in Figure 3.9.2, the tangent to the curve lies close to the curve near the point of tangency. For a brief interval to either side, the y-values along the tangent line give good approximations to the y-values on the curve.

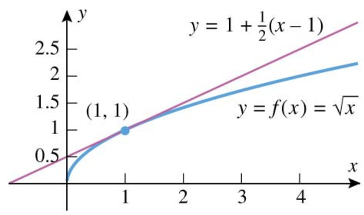

Figure 3.9.2

Definition 9.3. Linearization: If f is differentiable at $x = c$ then the approximating function

$$L(x) = f(c) + f'(c)(x - c)$$

is the **linearization** of f at c. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at c. The point $x = c$ is the **center of the approximation**.

- The approximation formula can also be expressed in terms of the increment $\Delta x = x - c$ as

$$f(c + \Delta x) \approx f(c) + f'(c)\Delta x.$$

- A linear approximation normally loses accuracy away from its center. If the graph of f has a pronounced "bend" at c, then we can expect that the accuracy of the local linear approximation of f at c will decrease rapidly as we progress away from c.
- The local linear approximation of f at x_0 has the property that its value and the value of its first derivative match those of f at c.

Example 9.4. Find the local linear approximation of $f(x) = \sqrt{x}$ at $x = 4$. Use that to approximate $\sqrt{4.1}$.
Solution:

9.2 Taylor Polynomials: Approximation by Higher-Degree Polynomials

We may try to improve on the accuracy of a local linear approximation by using higher-degree polynomials. Given a function f that can be differentiated n times at $x = c$, we will use a polynomial p of degree n with the property that the value of p and the values of its first n derivatives match those of f at c

Definition 9.5. If f can be differentiated n times at $x = c$, then we define the **The nth Taylor Polynomial for f about** $x = c$ to be

$$p_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + ... + \frac{f^{(n)}(c)}{n!}(x - c)^n,$$

where we use the notation $f^{(k)}(c)$ to denote the kth derivative of f at $x = c$.

- When $c = 0$, the Taylor polynomial $p_n(x)$ becomes the n**th Maclaurin polynomial**:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n$$

- The sigma notation for Taylor polynomial:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

where we make the convention that $f^{(0)}(c)$ denote $f(c)$.

- Local linear approximation: $f(x) \approx p_1(x)$.
- Local quadratic approximation: $f(x) \approx p_2(x)$.

Example 9.6. Find the Maclaurin polynomials p_0, p_1, p_2, p_3, and p_n for e^x.
Solution:

Example 9.7. Find the nth Maclaurin polynomial for $f(x) = \sin(x)$.
Solution:

Example 9.8. Find the nth Taylor polynomial for $f(x) = 1/x$ about $x = 1$.
Solution:

Example 9.9. Construct the Taylor polynomial of degree 4 approximating the function $\ln x$ for x near 1.
Solution:

Example 9.10. The function $f(x)$ is approximated near $x = 0$ by the second degree Taylor polynomial

$$P_2(x) = 5 - 7x + 9x^2$$

Give the value of

(a) $f(0)$

(b) $f'(0)$

(c) $f''(0)$

10 Infinite Sequences and Series

10.1 Sequences

Definition 10.1. An **infinite sequence** of numbers, or more simply a **sequence**, is a function whose domain is the set of positive integers.

- Stated informally, a sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

  in a given order.
- The dots are used to indicate that the sequence continues indefinitely.
- Each of a_1, a_2, a_3 and so on represents a number. These are the **terms** of the sequence.
- Sometimes the expression $\{a_n\}_{n=1}^{\infty}$ is used to describe the sequence.
  - The letter n is called the **index** for the sequence.
  - It is not essential to use n for the index. We might view the general term of the sequence $a_1, a_2, a_3, \ldots$ to be the kth term, in which case we would denote this sequence as $\{a_k\}_{k=1}^{+\infty}$.
  - It is not essential to start the index at 1.
  - When the starting value for the index of a sequence is not relevant to the discussion, it is common to use a notation such as $\{a_n\}$ in which there is no reference to the starting value of n.
- Some sequences have no simple formula. For example, the sequence $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ gives the first n digits of π.

10.2. There are two ways to represent sequences graphically.

(a) Mark the first few points on the real axis.

(b) Plot the function defining the sequence on integer inputs. The graph then consists of some points in the xy-plane, located at $(1, a_1)$, $(2, a_2)$, ..., (n, a_n), ...

Example 10.3. Represent the sequence $a_n = \frac{1}{n}$ graphically.
Solution:

10.2 Limit of a Sequence

Definition 10.4. Stated informally, we say that a sequence $\{a_n\}$ approaches a limit L if the terms in the sequence eventually become arbitrarily close to L. More formally, a sequence $\{a_n\}$ is said to **converge** to the **limit** L if given any $\varepsilon > 0$, there is a positive integer N such that $|a_n - L| < \varepsilon$ for $n \geq N$ (Figure 10.1.3). In this case we write

$$\lim_{n \to +\infty} a_n = L,$$

or simply $a_n \to L$. A sequence that does not converge to some finite limit is said to **diverge**.

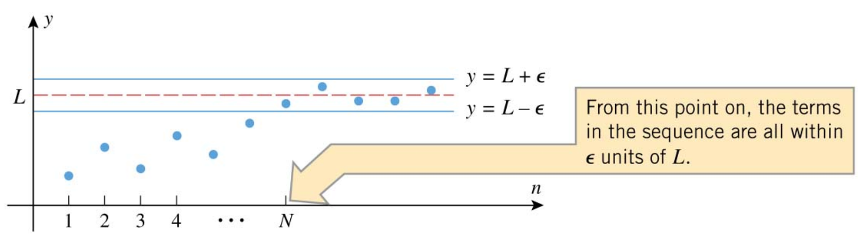

Figure 10.1.3

Example 10.5. Show that the sequence $\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.
Solution: Suppose the sequence converges to some number L. By choosing $\varepsilon = 1/2$ in the definition of the limit, we must find an integer N such that all terms a_n of the sequence with index n larger than N must lie within of L. Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\varepsilon = 1/2$ of L. It follows that L must satisfy $|L - 1| < 1/2$ or equivalently, $1/2 < L < 3/2$. Likewise, the number -1 appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L - (-1)| < 1/2$, or equivalently $-3/2 < L < -1/2$. But the number L cannot lie in both of the intervals $(1/2, 3/2)$ and $(-3/2, -1/2)$ because they have no overlap. Therefore, no such limit L exists and so the sequence diverges.

10.6. By definition, if $\lim_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} |a_n| = 0$.

Proof. The same n which works for $|a_n - 0| < \varepsilon$ would also work for $||a_n| - 0| < \varepsilon$.

□

10.7. To calculate the limit of a sequence, we can use what we know about the limits of functions.

- If we know that $\lim_{x \to \infty} f(x) = L$, then the limit of the sequence defined by $a_n = f(n)$ is also L.

- Because $\lim_{x\to\infty} \frac{1}{x} = 0$, we have $\lim_{n\to\infty} \frac{1}{n} = 0$.
- Because $\lim_{x\to+\infty} \frac{x}{x+1} = 1$, we have $\lim_{n\to+\infty} \frac{n}{n+1} = 1$

- When $\lim_{x\to\infty} f(x)$ does not exist, the sequence may converge or diverge.
  - We know that $\lim_{x\to\infty} \cos(2\pi x)$ DNE. However, the sequence $a_n = \cos(2\pi n) \equiv 1$ converges to the value 1.
  - We know that $\lim_{x\to\infty} \cos(\pi x)$ DNE. However, the sequence $a_n = \cos(\pi n) = (-1)^n$ diverges because it oscillates between -1 and 1.

Theorem 10.8. Squeezing Theorem for Sequences: Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that

$$a_n \le b_n \le c_n \quad \text{for all values of } n \text{ beyond some index } N.$$

If the sequence $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to \infty$, then $\{b_n\}$ also has the limit L as $n \to \infty$.

- This theorem is useful for finding limits of sequences that cannot be obtained directly.

Example 10.9. Find the limit of the sequence $\left\{\frac{n!}{n^n}\right\}_{n=1}^{+\infty}$.
Solution: Rewrite the general term as

$$a_n = \frac{1}{n}\left(\frac{2\cdot 3\cdots n}{n\cdot n\cdots n}\right).$$

Observe that

$$0 \le a_n \le \frac{1}{n}.$$

Example 10.10. Find the limit of the sequence $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$.
Solution:

Theorem 10.11. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. $-|a_n| \le a_n \le |a_n|$. □

- Combining this result with 10.6, we then know that $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.

Definition 10.12. Diverges to Infinity: The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$ then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty.$$

- A sequence may diverge without diverging to infinity or negative infinity.

Theorem 10.13. The Continuous Function Theorem for Sequences: Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Definition 10.14. If discarding finitely many terms from the beginning of a sequence produces a sequence with a certain property, the the original sequence is said to have that property **eventually**.