Sirindhorn International Institute of Technology
Thammasat University
Department of Common and Graduate Studies

MAS 116: Lecture Notes 7

Semester: 3/2008
Instructors: Dr. Prapun Suksompong

6 Applications of the Definite Integral

Riemann sums and definite integrals have applications that extend far beyond the area problem. The required calculations can all be approached by the same procedure that we used to find areas – (1) breaking the required calculation into "small parts," (2) making an approximation for each part, (3) adding the approximations from the parts to produce a Riemann sum that approximates the entire quantity to be calculated, and then (4) taking the limit of the Riemann sums to produce an exact result.

6.1 Area Between Two Curves

Theorem 6.1. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ and $y = g(x)$ from a to b is

$$\int_a^b [f(x) - g(x)]\, dx.$$

- The condition $f(x) \geq g(x)$ throughout $[a, b]$ means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross. Therefore, the above integration can be written as
$$\int_a^b [\text{ top } - \text{ bottom }] dx$$

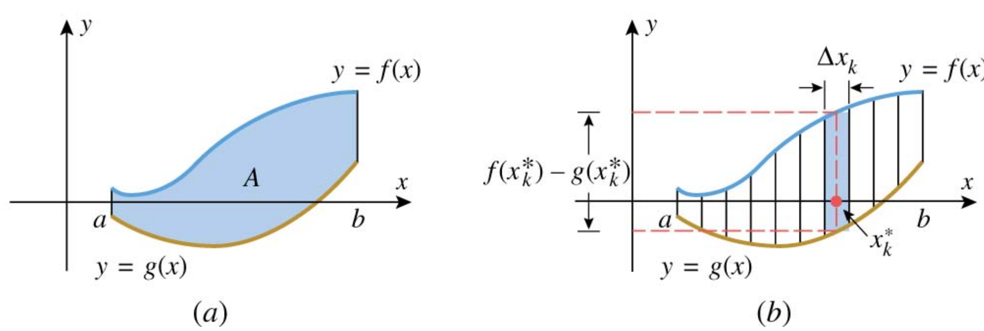


Figure 6.1.3

Example 6.2. Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$ as shown in Figure 6.1.4.

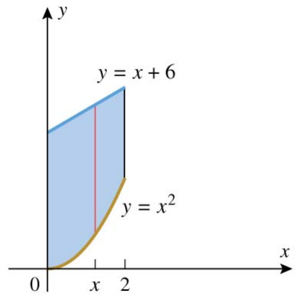


Figure 6.1.4

Solution:

6.3. It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments. When that occurs, you will have to determine the points of intersection to obtain the limits of integration.

Example 6.4. Find the area of the region that is enclosed between the curves $y = x + 6$ and $y = x^2$ as shown in Figure 6.1.6.

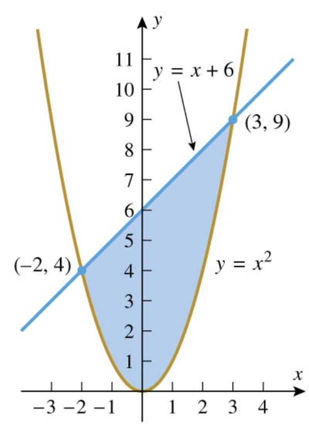


Figure 6.1.6

Solution:

6.2 Finding Volumes by Slicing

6.5. A **right cylinder** is a solid that is generated when a plane region is translated along a line or axis that is perpendicular to the region (Figure 6.2.3).

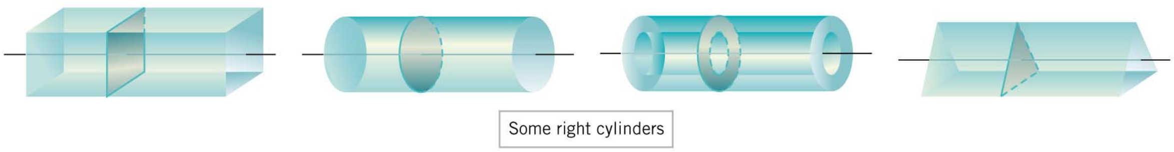


Figure 6.2.3

If a right cylinder is generated by translating a region of area A through a distance h the h is called the **height** (or sometimes the **width**) of the cylinder, and the volume V of the cylinder is defiend to be

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}]$$

(Figure 6.2.4).

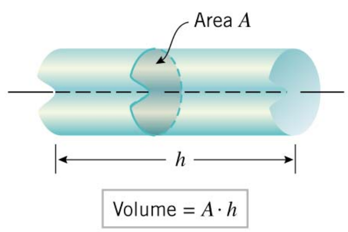


Figure 6.2.4

Theorem 6.6. Finding Volumes by the Method of Slicing: The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.

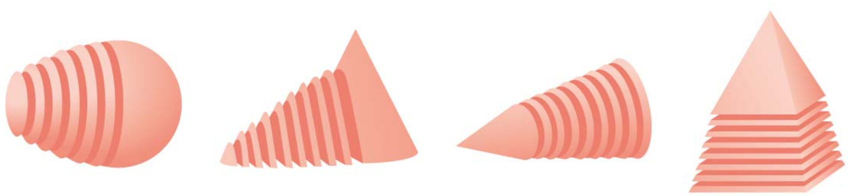


Figure 6.2.1

(a) Consider a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ an $x = b$. If for each x in $[a, b]$, the cross-sectional area of of the solid perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)dx. \tag{17}$$

- The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis.
- Volume of the kth slab $\approx V_k = A(x_k)\Delta x_k$.

(b) Consider a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ an $y = d$. If for each y in $[c, d]$, the cross-sectional area of of the solid perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)dy.$$

Example 6.7. Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with slides of length a.
Solution:

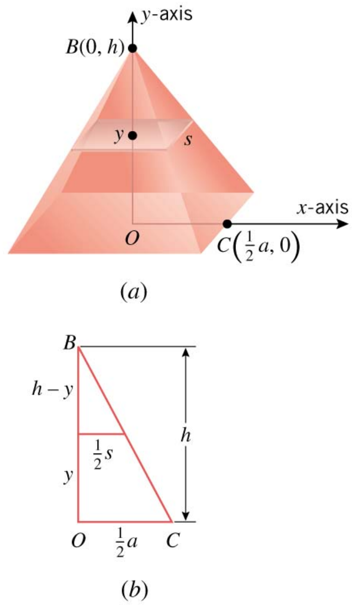


Figure 6.2.7

6.8. Cavalieris Principle: Cavalieris principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume. This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

Definition 6.9. The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**.

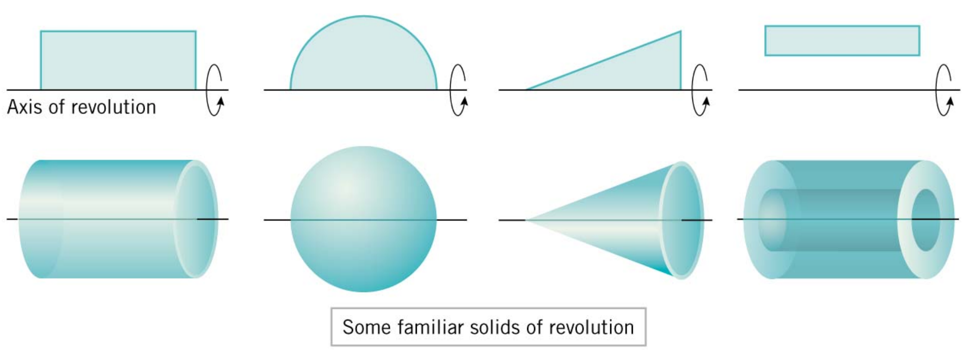


Figure 6.2.8

Theorem 6.10. A Solid of Revolution – Rotation about the x-Axis: Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. The volume of the solid of revolution that is generated by revolving the region R about the x-axis is given by

$$V = \int_a^b \pi \left[f(x)\right]^2 dx. \tag{18}$$

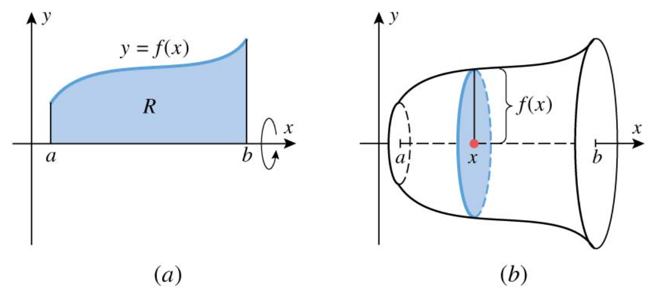


Figure 6.2.9

- Because the cross sections are disk shaped, the application of this formula is called the **method of disks** or **disk method**.
- Formula (18) follows directly from (17) by setting $A(x) = \pi\,[f(x)]^2$.

Example 6.11. Find the volume of a sphere of radius r.
Solution: As indicated in Figure 6.2.11, a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x-axis and

$$x^2 + y^2 = r^2$$

about the x-axis.

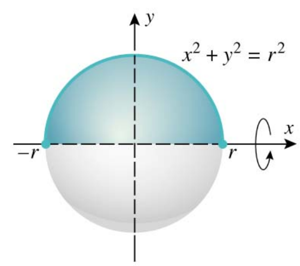


Figure 6.2.11

$$V = \int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2 dx = \frac{4}{3}\pi r^3.$$

Example 6.12. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 6.2.10).
Solution:

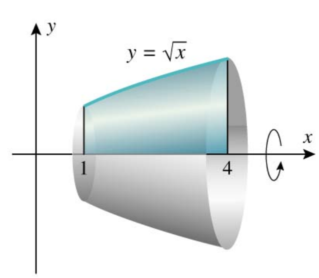


Figure 6.2.10

6.3 Volumes by Cylindrical Shells

Some volume problems are difficult to handle by the slicing method discussed earlier. In this section we will develop another method for finding volumes.

6.13. A **cylindrical shell** is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2).

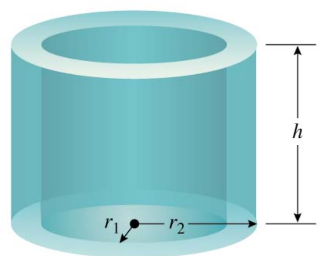


Figure 6.3.2

Theorem 6.14. Volume by Cylindrical Shells about the y-axis: Let f be continuous and nonnegative on $[a, b]$ $(0 \leq a < b)$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the y axis is given by

$$V = \int_a^b 2\pi x f(x) dx. \tag{19}$$

6.15. Observe that we slice through the solid using circular cylinders of increasing radii, like cookie cutters (Figure 6.3.3b). We slice straight down through the solid perpendicular to the x-axis, with the axis of the cylinder parallel to the y-axis. The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx (Figure6.3.7).

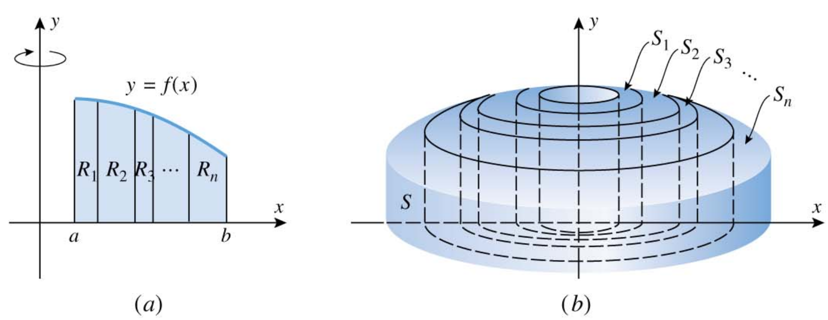


Figure 6.3.3

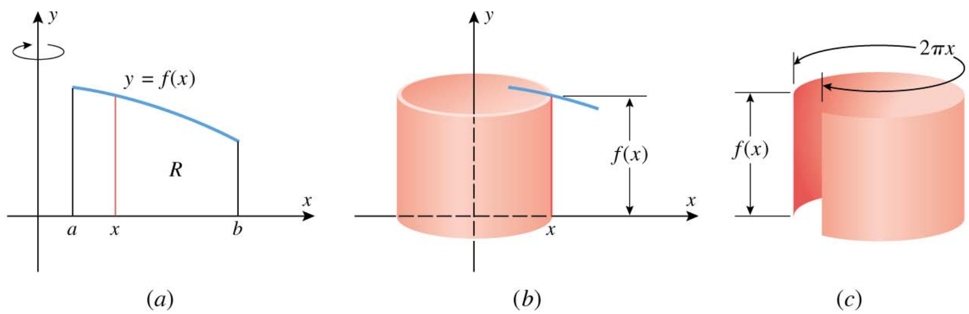


Figure 6.3.7

Another (and more rigorous) way to see that (19) works is to consider the formula of the volume of a cylindrical shell directly. The volume of a cylindrical shell with inner radius r_1, outer radius r_2, and height h (Figure 6.2.2) is given by

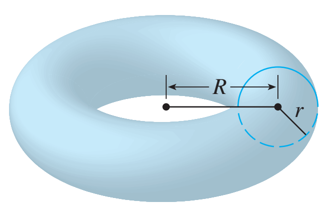
$$V = \left(\pi r_2^2 - \pi r_1^2\right) h = 2\pi \times \underbrace{\frac{r_1 + r_2}{2}}_{\text{average radius}} \times h \times \underbrace{(r_2 - r_1)}_{\text{thickness}}$$

The corresponding Riemann sum that approximates the volume is given by

$$V \approx \sum_{k=1}^{n} 2\pi c_k f(c_k) \Delta x_k$$

where we choose c_k to be the midpoint of the interval $[x_{k-1}, x_k]$: $(x_k + x_{k-1})/2$.

Example 6.16. Use cylindrical shells to find the volume of the solid torus (a donut-shaped solid with radii r and R) shown in the figure below).



7 Inverse Trigonometric Function

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations.

7.1. The six basic trigonometric functions are not one-to-one (their values repeat periodically). However we can restrict their domains to intervals on which they are one-to-one. Figure 7.7.1 and Table 7.7.1 summarize the basic properties of three important inverse trigonometric functions.

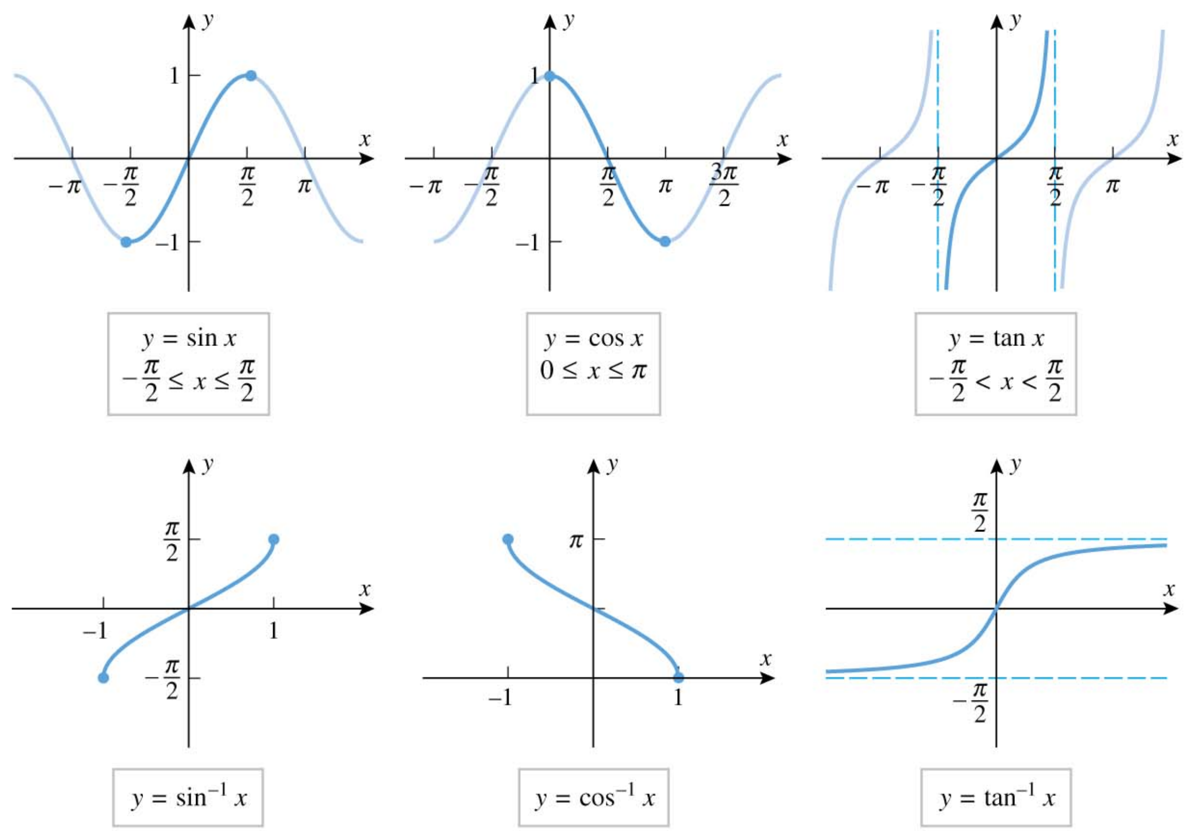


Figure 7.7.1

Table 7.7.1

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$

- The "-1" in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the reciprocal of $\sin x$ is $(\sin x)^{-1} = \csc x$.

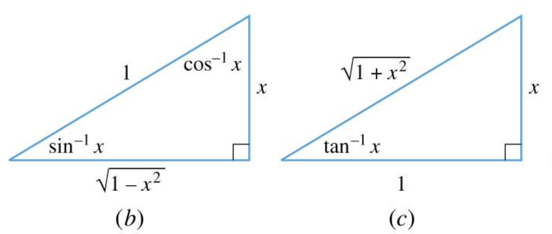
- Observe that the inverse sine and inverse tangent are odd functions:
$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad \text{and} \quad \tan^{-1}(-x) = -\tan^{-1}(x).$$
However,
$$\cos^{-1}(-x) = \pi - \cos^{-1} x.$$
- The "Arc" in Arc Sine and Arc Cosine: If $x = \sin\theta$ then, in addition to being the angle whose sine is x, θ is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call θ "the arc whose sine is x."

7.2. Inverse Function-Inverse Cofunction Identities [3][p 527]:

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$
$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$
$$\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$$

Therefore, the derivative of an inverse cofunction is the same as the derivative of a corresponding inverse function with an extra negative sign.

7.3. If we interpret $\sin^{-1} x$ (or $\cos^{-1} x$, or $\tan^{-1} x$) as an angle θ in radian measure whose sine (or cosine, or tangent, respectively) is x, and if that angle is between 0 and $\pi/2$ (or equivalently, x is positive), then we can represent θ geometrically as an angle in a right triangle as shown below.



Such triangle reveals useful identities:

$$\cos\left(\sin^{-1}x\right) = \sin\left(\cos^{-1}x\right) = \sqrt{1-x^2} \tag{20}$$
$$\tan\left(\sin^{-1}x\right) = \frac{1}{\tan\left(\cos^{-1}x\right)} = \frac{x}{\sqrt{1-x^2}} \tag{21}$$
$$\sin\left(\tan^{-1}x\right) = \frac{x}{\sqrt{1+x^2}} \tag{22}$$
$$\cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+x^2}} \tag{23}$$

It turns out that the identities above are also valid when x is negative. For example, when $-1 < x < 0$, we know, from (20), that
$$\cos\left(\sin^{-1}(-x)\right) = \sqrt{1-(-x)^2}$$
because $0 < -x < 1$. The we can simplify the LHS to be
$$\cos\left(\sin^{-1}(-x)\right) = \cos\left(-\sin^{-1}(x)\right) = \cos\left(\sin^{-1}(x)\right).$$
So, we get back (20) but now for $-1 < x < 0$. Note that we also have to directly check that the identity is valid for $x = 0$ and at the two endpoints. Similar reasoning extends the rest of the identities.

Note that the triangle technique does not always produce the most general form of an identity. For example, it does not give $|\cdot|$ in
$$\sin\left(\sec^{-1}x\right) = \frac{\sqrt{x^2-1}}{|x|}.$$

Example 7.4. A derivative formula for $\sin^{-1} x$ can be obtained via implicit differentiation (Section 3.9) or the Formula (14) for the differentiation of inverse function. To use implicit differentiation, we rewrite the equation $y = \sin^{-1} x$ as $x = \sin y$ and differentiate with respect to x. We then obtain

$$\begin{aligned} \frac{d}{dx}[x] &= \frac{d}{dx}[\sin y] \\ 1 &= \cos y \cdot \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} = \frac{1}{\cos\left(\sin^{-1} x\right)}, \end{aligned}$$

which we can alternatively obtain directly via Formula (14). This derivative formula can be simplified using (20). This yields

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

The method used to derive this formula can be used to obtain derivative formulas for the remaining inverse trig. functions.

Theorem 7.5. Derivatives of the inverse trigonometric functions:

$$\begin{aligned} \frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}}, \quad (|x|<1) \\ \frac{d}{dx}\cos^{-1} x &= -\frac{1}{\sqrt{1-x^2}}, \quad (|x|<1) \\ \frac{d}{dx}\tan^{-1} x &= \frac{1}{1+x^2}. \end{aligned}$$

Theorem 7.6. Integration Formulas:

$$\begin{aligned} \int \frac{1}{\sqrt{a^2-x^2}} dx &= \sin^{-1}\left(\frac{x}{a}\right) + C, \quad (x^2 < a^2) \\ \int \frac{1}{a^2+x^2} dx &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C. \end{aligned}$$

- The formulas are readily verified by differentiating the functions on the right-hand sides.

Example 7.7. Completing the Square:

$$\int \frac{1}{\sqrt{8x-x^2}} dx = \int \frac{1}{\sqrt{16-(x-4)^2}} dx \underset{u=x-4}{=} \int \frac{1}{\sqrt{4^2-u^2}} du = \sin^{-1}\left(\frac{u}{4}\right) + C = \sin^{-1}\left(\frac{x-4}{4}\right) + C$$

8 Additional Integration Techniques

8.1 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)\, g(x) dx.$$

8.1. ***Integration by Parts***:

$$\int f(x)\, g'(x) dx = f(x)\, g(x) - \int f'(x)\, g(x) dx. \tag{24}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)dx$ and $dv = g'(x)dx$. Using the Substitution Rule, the integration by parts formula becomes

$$\int udv = uv - \int vdu \tag{25}$$

- To see (24), start with the product rule: $(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$. Then, integrate both sides.
- The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate then the original. In other words, the goal of integration by parts is to go from an integral $\int udv$ that we dont see how to evaluate to an integral $\int vdu$ that we can evaluate.
  - In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice.
- Note that when we calculate v from dv, we can use *any* of the antiderivative. In other words, we may put in $v + C$ instead of v in (25). Had we included this constant of integration C in (25), it would have eventually dropped out. This is always the case in integration by parts.

8.2. We have four possible choices when applying integration by parts to an integral of the form

$$\int f(x)\, g(x) dx :$$

(a) Let $u = 1$ and $dv = f(x)g(x)dx$. This choice won't do because we have to find $v = \int f(x)g(x)dx$ which is the same as the original integral.

(b) Let $u = f(x)$ and $dv = g(x)dx$

(c) Let $u = g(x)$ and $dv = f(x)dx$

(d) Let $u = f(x)g(x)$ and $dv = dx$

Therefore, we have three useful choices. Remember that we must be able to readily integrate dv to get v in order to obtain the right side of the formula (25). If the new integral on the right side is more complex than the original one, try a different choice for u and dv.

Example 8.3. Use integration by parts to evaluate $\int x \cos x dx$.
Solution:

8.4. Evaluating Definite Integrals by Parts: For definite integrals, the formula corresponding to (24) is

$$\int_a^b f(x)\, g'(x) dx = f(x)\, g(x)]_a^b - \int_a^b f'(x)\, g(x) dx. \tag{26}$$

The corresponding u and v notation is

$$\int_a^b udv = uv]_a^b - \int_a^b vdu \tag{27}$$

- It is important to keep in mind that the variables u and v in this formula are functions of x and that the limits of integration in (27) are limits on the variable x. Sometimes it is helpful to emphasize this by writing (27) as

$$\int_{x=a}^{b} u dv = uv]_{x=a}^{b} - \int_{x=a}^{b} v du \tag{28}$$

Example 8.5. Evaluate the following integrals.

(a) $\int xe^x dx$
Solution:

(b) $\int_2^3 \ln(t) dt$
Solution:

(c) $\int_0^1 \tan^{-1}(s) ds$
Solution:

Example 8.6. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$

8.7. Tabular Integration by Parts: Repeated application of integration by parts gives

$$\int f(x) g(x) dx = f(x) G_1(x) + \sum_{i=1}^{n-1} (-1)^i f^{(i)}(x) G_{i+1}(x) + (-1)^n \int f^{(n)}(x) G_n(x) dx \tag{29}$$

where $f^{(i)}(x) = \frac{d^i}{dx^i} f(x)$, $G_1(x) = \int g(x) dx$, and $G_{i+1}(x) = \int G_i(x) dx$.

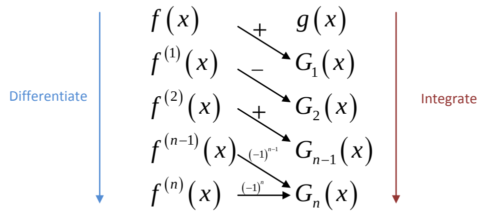
- To see this, note that

$$\int f(x) g(x) dx = f(x) G_1(x) - \int f'(x) G_1(x) dx,$$

and

$$\int f^{(n)}(x) G_n(x) dx = f^{(n)}(x) G_{n+1}(x) - \int f^{(n+1)}(x) G_{n+1}(x) dx.$$

A convenient method for organizing the computations into two columns is called tabular integration by parts.



Example 8.8. Evaluate $\int x^2 \sin(x)dx$

Solution:

Integrals like the one in the next example requires two integrations by parts, followed by solving for the unknown integral.

Example 8.9. Evaluate $\int e^x \sin(x)dx$
Solution:

Example 8.10. If n is a positive integer,

$$\int x^n e^{ax} dx = \frac{e^{ax}}{a} \sum_{k=0}^{n} \frac{(-1)^k \, n!}{a^k \, (n-k)!} x^{n-k}.$$

(a) $n = 1 : \frac{e^{ax}}{a} \left(x - \frac{1}{a} \right)$

(b) $n = 2 : \frac{e^{ax}}{a} \left(x^2 - \frac{2}{a} x + \frac{2}{a^2} \right)$

Example 8.11. Let f be twice differentiable with $f(0) = 6$, $f(1) = 5$, and $f'(1) = 2$. Evaluate the integral $\int_0^1 x f''(x) dx$
Solution:

8.2 Methods for Approaching Integration Problem

8.12. There are three basic approaches for evaluating unfamiliar integrals:

(a) Technology: Computer Algebra Systems (CAS) such as Mathematica, Maple, and Derive can be used to evaluate an integral, if such a system is available.

(b) Tables: Prior to the development of CAS, scientists relied heavily on tables to evaluate difficult integrals arising in applications. Such table were compiled over many years, incorporating the skills and experience of many people. Extensive tables appear in compilations such as *CRC Mathematical Tables*, which contain thousands of integrals.

(c) Transformation Methods: Transformation methods are methods for converting unfamiliar integrals into familiar integrals.

None of these methods is perfect.

(a) CAS often encounter integrals that they cannot evaluate and they sometimes produce answers that are unnecessarily complicated.

(b) Tables are not exhaustive and hence may not include a particular integral of interest.

(c) Transformation methods rely on human ingenuity that may prove to be inadequate in difficult problems.

8.3 Improper Integrals

Our main objective in this section is to extend the concept of a definite integral to allow for (1) infinite interval of integration and (2) integrands with vertical asymptotes within the interval of integration.

Definition 8.13. Integrals over Infinite Intervals: Integrals with infinite limits of integration are **improper integrals of Type I**.

(a) If $f(x)$ is continuous on $[a, \infty)$ then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

(b) If $f(x)$ is continuous on $(-\infty, b]$ then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**, and it is not assigned a value.

Example 8.14. Evaluate $\int_1^\infty \frac{1}{x^2}dx$.
Solution:

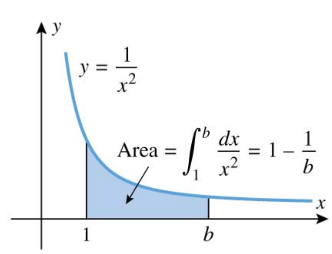


Figure 8.8.2

Example 8.15. Evaluate $\int_0^\infty \frac{1}{1+x^2}dx$ and $\int_{-\infty}^0 \frac{1}{1+x^2}dx$.
Solution:

Example 8.16. Gamma function: $\Gamma(n) = \int\limits_0^\infty x^{n-1}e^{-x}dx = (n-1)! \quad ; \quad n > 0.$

- Integration by parts show that $\Gamma(n+1) = n\Gamma(n)$.

Definition 8.17. Integrals over Infinite Intervals (con't): If $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

where c is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.

- It can be shown that the choice of c is unimportant.

8.18. Any of the integrals in Definitions 8.13 and 8.17 can be interpreted as an area if $f \geq 0$ on the interval of integration. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 8.19. Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$.
Solution:

Definition 8.20. Integrands with Vertical Asymptotes (Infinite Discontinuity): Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

- If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx.$$

- If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b then

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the value of the improper integral. If the limit does not exist, the integral **diverges**.

- If $f(x)$ is continuous on the interval $[a,b]$ except for a discontinuity at a point c in (a,b), then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

The integral on the left side of the equation **converges** if both integrals on the right side converge; otherwise it **diverges**.

Example 8.21. Evaluate the following integrals

(a) $\int_0^1 \frac{1}{x^2} dx$
Solution:

(b) $\int_0^1 \frac{1}{\sqrt{x}} dx$
Solution:

Example 8.22. The function $1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form x^α:

$$\int_0^1 t^\alpha dt = \begin{cases} \frac{1}{\alpha+1}, & \alpha > -1 \\ \infty, & \alpha \leq -1 \end{cases}$$

and

$$\int_1^\infty t^\alpha dt = \begin{cases} \frac{1}{\alpha+1}, & \alpha < -1 \\ \infty, & \alpha \geq -1. \end{cases}$$

In fact, $\int_0^1 \frac{1}{t} dt = \int_1^\infty \frac{1}{t} dt = \infty$.