Sirindhorn International Institute of Technology
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MAS 116: Lecture Notes 6

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5 Integration

5.1 Antiderivative and Indefinite Integral

Definition 5.1. A function F is called an **antiderivative** of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

The process of finding antiderivatives is called **antidifferentiation** or **integration**.

Example 5.2. Given a function $f(x) = x^2$, can we find some function $F(x)$ whose derivative is x^2?. Notice that if we let

$$F(x) = \frac{x^3}{3}$$

then,

$$F'(x) = \frac{1}{3}3x^2 = x^2$$

Are there any function F whose derivative is x^2? If we add any constant C to $F(x) = \frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f.

5.3. In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. It is then reasonable to ask if there are antiderivatives of a function f that cannot be obtained by adding some constant to a known antiderivative F. The answer is *no*.

Theorem 5.4. If $F'(x) = G'(x)$ on an interval I, then on I,

$$F(x) = G(x) + C$$

for some constant C.

Definition 5.5. The *collection* of all antiderivatives of a function $f(x)$ is called the **indefinite integral** and is denoted by

$$\int f(x)dx$$

If we know one function $F(x)$ for which $F'(x) = f(x)$, then

$$\int f(x)dx = F(x) + C$$

where C is any arbitrary constant. This constant is called the **constant of integration**. We refer the symbol $\int$ as the **integral sign** and $f(x)$ as the **integrand**.

- Graphs of antiderivatives of a function f are called **integral curves** of f.

5.6. Many basic integration formulas can be obtained directly from their companion differentiation formulas:

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

(b) $\int e^x dx = e^x + C$

(c) $\int \frac{1}{x} dx = \ln|x| + C$

(d) $\int \cos(x) dx = \sin(x) + C$

(e) $\int \sin(x) dx = -\cos(x) + C$

5.7. Properties of the Indefinite Integral:

(a) $\int kf(x)dx = k\int f(x)dx$

(b) $\int[f(x) \pm g(x)]dx = \int f(x) \pm \int g(x)dx$

(c) **Linearity of Indefinite Integral**:

$$\int af(x) + bg(x)dx = a\int f(x)dx + b\int g(x)dx$$

- These equations must be applied carefully to avoid errors and unnecessary complexities arising from the constants of integration.

Example 5.8. Evaluate the following integrals.

(a) $\int x^5 dx$
Solution:

(b) $\int \frac{1}{\sqrt{x}} dx$
Solution:

(c) $\int \pi \sqrt[5]{u^3} + 1 du$
Solution:

(d) $\int \frac{2}{x} + \frac{x^3}{\sqrt{2}} dx$
Solution:

5.9. Sometimes it is useful to rewrite an integrand in a different form before performing the integration.

Example 5.10. Evaluate the following integrals.

(a) $\int \frac{x+1}{x} dx$
Solution:

(b) $\int \sqrt{t}(t+1)dt$
Solution:

(c) $\int \frac{e^{-u}+1}{e^{-u}} du$
Solution:

5.11. Integration by Substitution: The substitution technique is based on reversing the chain rule. By chain rule we have

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

If F is an antiderivative of f, then we can write

$$\frac{d}{dx}[F(g(x))] = f(g(x))g'(x).$$

Therefore

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

5.12. Method of u-Substitution:

(a) Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \quad du = g'(x)dx$$

produces an integral that is expressed entirely in terms of u and du.

(b) Try to evaluate the resulting integral in terms of u.

(c) Replace u by $g(x)$.

Example 5.13. Evaluate the following integrals

(a) $\int 2x(x^2+1)^{99}dx$
Solution:

(b) $\int \frac{3x^2}{\sqrt{x^3+3}}dx$
Solution:

(c) $\int \frac{2\ln x}{x}dx$
Solution:

(d) $\int \frac{x+1}{x^2+2x}dx$
Solution:

(e) $\int \frac{x+1}{x^2+2x} dx$
Solution:

(f) $\int x e^{x^2-1} dx$
Solution:

(g) $\int \cos^3 x \sin x dx$
Solution:

Example 5.14. Evaluate the following integrals

(a) $\int x^2 \sqrt{x-1} dx$
Solution:

(b) $\int \cos^3 x dx$
Solution:

Example 5.15. The Integrals of $\sin^2 x$ and $\cos^2 x$

- $\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$
- $\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Example 5.16.

$$\int f(ax+b)dx \overset{u=ax+b}{=} \int f(u)\frac{1}{a}du = \frac{1}{a}F(u) = \frac{1}{a}F(ax+b).$$

In particular,

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$$
$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

5.17. Slope Field and Differential Equation: Finding antiderivatives of $f(x)$ is the same as finding a function $F(x)$ such that $y = F(x)$ satisfies the **differential equation**

$$\frac{dy}{dx} = f(x). \tag{13}$$

If we interpret dy/dx as the slope of a tangent line, then at a point (x, y) on an integral curve of Equation (13), the slope of the tangent line is $f(x)$. Note that these slopes can be obtained without actually solving the differential equation.

A geometric description of the integral curves of a differential equation (13) can be obtained by

(a) choosing a rectangular grid of points in the xy-plane,

(b) calculating the slopes of the tangent lines to the integral curves at the gridpoints, and

(c) drawing small portions of the tangent lines through those points.

The resulting picture, which is called a **slope field** or **direction field** for the equation, shows the "direction" of the integral curves at the grid points. With sufficiently many gridpoints it is often possible to visualize the integral curves themselves.

Example 5.18. Figure 5.2.3a shows a slope field for the differential equation $dy/dx = x^2$, and figure 5.2.3b shows that same field with the integral curves imposed on it.

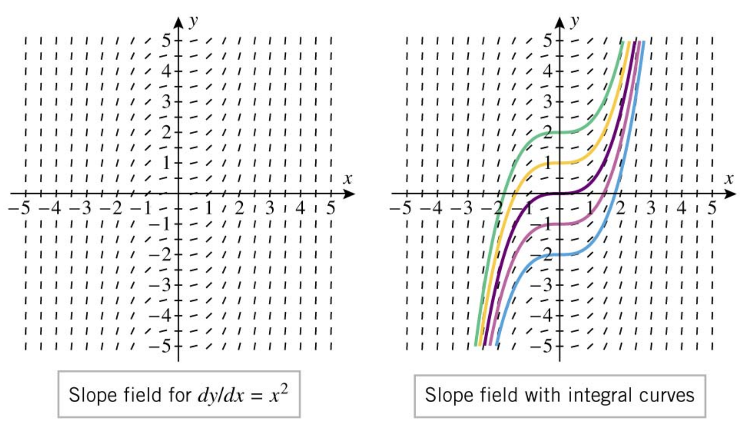

Figure 5.2.3 (a) (b)

5.2 Definite Integral and Riemann sum

We will develop the theory of the definite integral in the setting of area, where it most clearly reveals its nature. The idea behind definite integration is that we can effectively compute many quantities by breaking them into small pieces, and then summing the contributions from each small part.

5.19. Numerical Approximations of Area

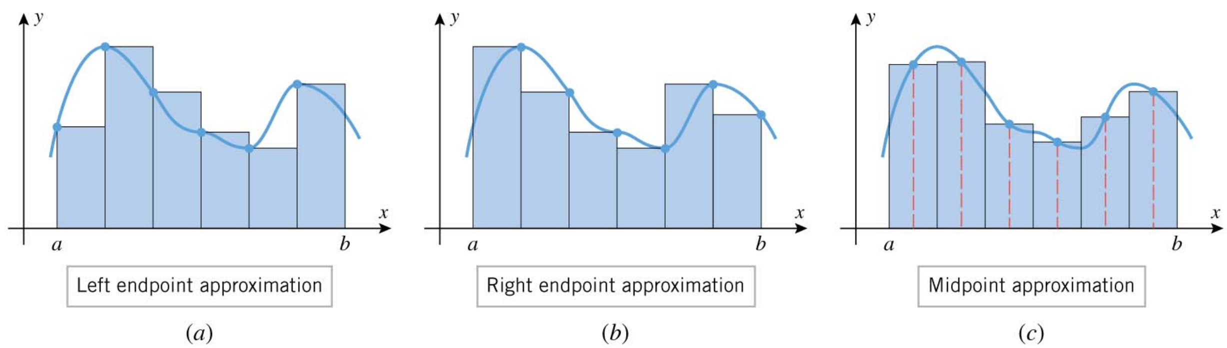

Figure 5.4.9

Example 5.20. Find the left endpoint approximation and the left endpoint approximation of the area under the curve $f(x) = x^2$ over the interval $[0, 1]$ with $n = 5$. Illustrate each part with a graph of f that includes the rectangles whose areas are represented in the sum.
Solution:

Example 5.21. Find the left endpoint approximation and the left endpoint approximation of the area under the curve $f(x) = \frac{1}{x}$ over the interval $[0, 1]$ with $n = 5$. Illustrate each part with a graph of f that

includes the rectangles whose areas are represented in the sum.
Solution:

5.22. Riemann Sums: We begin with an arbitrary function f defined on a closed interval $[a, b]$.

(a) Subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths (or lengths). To do so, we choose $n-1$ points $(x_1, x_2, \ldots, x_n)$ between a and b and satisfying

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we denote a by x_0 and b by x_n so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set P is called a **partition** of $[a, b]$.

- The partition P divides $[a, b]$ into n closed subintervals.

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

- The kth subinterval of P is $[x_{k-1}, x_k]$,for k an integer between 1 and n.
- The width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$.
  - If all n subintervals have equal width, then the common width Δx is equal to $(b-a)/n$.
    - $x_k = a + k\Delta x$.
  - We define the **norm** of a partition P, written $\|P\|$ to be the largest of all the subinterval widths.
    - If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width.

(b) In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k.

- In practice, the c_k are chosen in some systematic fashion, some common choices being
  (i) the left endpoint of each subinterval
  (ii) the right endpoint of each subinterval
  (iii) the midpoint of each subinterval
- If all n subintervals have equal width, then the left endpoint, right endpoint, and midpoint choices for the c_k are given by
  (i) Left endpoint: $c_k = a + (k-1)\Delta x$
  (ii) Right endpoint: $c_k = a + k\Delta x$
  (iii) Midpoint: $c_k = a + \left(k - \frac{1}{2}\right)\Delta x$

(c) Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$.

- These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on it if $f(c_k) = 0$.

(d) On each subinterval we form the product $f(c_k) \cdots \Delta x_k$.

- When $f(c_k) > 0$ the product $f(c_k) \cdots \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k.
- When $f(c_k) < 0$ the product $f(c_k) \cdots \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

(e) Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

- The sum S_P is called a **Riemann sum** for f on the interval $[a, b]$.
- There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subintervals.
- Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x-axis.
- Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy

Definition 5.23. The Definite Integral as a Limit of Riemann Sums: A function f is said to be **integrable** on a finite closed interval $[a, b]$ if the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points c_k in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

which is called the **definite integral** of f from a to b.

- The numbers a and b are called the **lower limit of the integration** and the **upper limit of integration**, respectively.

- $f(x)$ is called the **integrand**.
- Read: "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x."
- The limit is always taken as the norm of the partitions approaches zero and the number of subintervals goes to infinity.
- The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$int_a^b f(t)dt \quad \text{or} \quad int_a^b f(u)du \quad \text{instead of} \quad int_a^b f(x)dx.$$

No matter how we write the integral, it is still the same number, defined as a limit of Riemann sums. Since it does not matter what letter we use, the variable of integration is called a **dummy variable**.

Theorem 5.24. Continuous Function is Integrable: If a function f is continuous on an interval $[a,b]$, then its definite integral over $[a,b]$ exists.

- If the function f is continuous over the closed interval $[a,b]$, then no matter how we choose the partition P and the points c_k in its subintervals to construct a Riemann sum, a single limiting value is approached as the subinterval widths, controlled by the norm of the partition, approach zero.
- Functions that are not continuous may or may not be integrable.

5.25. For integrability to fail, a function needs to be sufficiently discontinuous so that the region between its graph and the x-axis cannot be approximated well by increasingly thin rectangles.

Theorem 5.26. Let f be a function that is defined on the finite closed interval $[a,b]$.

(a) If f has finitely many discontinuities in $[a,b]$ but is bounded on $[a,b]$, then f is integrable on $[a,b]$.

(b) If f is not bounded on $[a,b]$, then f is not integrable on $[a,b]$.

Definition 5.27.

(a) If a is in the domain of f, we define

$$\int_a^a f(x)dx = 0.$$

(b) If f is integrable on $[a,b]$, then we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

5.28. Properties of Definite Integrals: Suppose f and g are integrable on a closed interval containing the three points a, b, and c. Let k be any constant.

(a) kf, $f+g$, and $f-g$ are integrable on $[a,b]$

(b) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

(c) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)dx$

(d) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(e) If $f(x) \geq 0$ for all x in $[a,b]$, then $\int_a^b f(x)dx \geq 0$.

(f) If $f(x) \geq g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Definition 5.29. Area Under a Curve as a Definite Integral: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)dx.$$

If f is a continuous function on the interval $[a, b]$, then we define the total area between the curve $y = f(x)$ and the interval $[a, b]$ to be

$$\int_a^b |f(x)|dx.$$

5.30. In the simplest cases, definite integrals of continuous functions can be calculated using formulas from plane geometry to compute the areas.

Example 5.31. Evaluate

$$\int_0^3 4 - \sqrt{9 - x^2}dx$$

Solution:

Example 5.32. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} |x - 1| & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

(a) $\int_{-1}^0 f(x)dx$
Solution:

(b) $\int_0^1 f(x)dx$
Solution:

(c) $\int_{-2}^3 f(x)dx$
Solution:

Example 5.33. Given that

$$\begin{aligned} \int_1^4 f(x)dx &= 2 \\ \int_1^4 g(x)dx &= -1 \\ \int_4^0 f(x)dx &= 2 \end{aligned}$$

Find

(a) $\int_1^4 [f(x) - 2g(x)]dx$
Solution:

(b) $\int_0^1 f(x)dx$
Solution:

5.3 Fundamental Theorem of Calculus

Theorem 5.34. Fundamental Theorem of Calculus, Part 1: If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

- $\int_a^x f(t)dt$ is an antiderivative of f.
- In the textbook by Anton [1], this is labeled as "Part 2".

Example 5.35. Use the Fundamental Theorem of Calculus to find the following derivatives.

(a) $\frac{d}{dx} \int_0^x \sqrt{1+t^2}dt$
Solution:

(b) $\frac{d}{dt} \int_t^5 e^{-x^2} dx$
Solution:

(c) $\frac{d}{ds}\int_0^{s^3} \cos t^2 dt$
Solution:

(d) $\frac{d}{dx}\int_{e^x}^{x^2} \sqrt{1+t^2}dt$
Solution:

Theorem 5.36. Fundamental Theorem of Calculus, Part 2 (The Evaluation Theorem): If f is continuous at every point of $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

- The usual notation for $F(b) - F(a)$ is

$$F(x)]_a^b \quad \text{or} \quad [F(x)]_a^b$$

Example 5.37. Evaluate the following integrals:

(a) $\int_1^2 1 + 6x^2 dx$
Solution:

(b) $\int_0^\pi \sin(x) + e^x + x^2 dx$
Solution:

(c) $\int_0^1 \frac{x^2-2x}{x} dx$
Solution:

5.38. Evaluating Definite Integrals by Substitution: We have two choices.

(a) Evaluate the indefinite integral by substitution first: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

(b) Make the substitution directly in the definite integral: Transform the integral and evaluate the transformed integral with the transformed limits.

- See Theorem 5.39 below.

Theorem 5.39. If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 5.40. Evaluate the following integrals.

(a) $\int_0^1 \sin(\pi x)dx$
Solution:

(b) $\int_0^{\pi/4} \tan\theta d\theta$
Solution:

(c) $\int_2^3 \frac{z}{z^2-1}dz$
Solution:

(d) $\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}}dt$
Solution:

Example 5.41. Suppose

$$\int_0^1 f(t)dt = 3$$

Calculate the following

(a) $\int_0^{0.5} f(2t)dt$
Solution:

(b) $\int_0^1 f(1-t)dt$
Solution:

(c) $\int_1^{1.5} f(3-2t)dt$
Solution: