Sirindhorn International Institute of Technology
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MAS 116: Lecture Notes 6

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5 Integration

5.1 The Indefinite Integral

Definition 5.1. A function F is called an **antiderivative** of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

The process of finding antiderivatives is called **antidifferentiation** or **integration**.

Example 5.2. Given a function $f(x) = x^2$, can we find some function $F(x)$ whose derivative is x^2?. Notice that if we let

$$F(x) = \frac{x^3}{3}$$

then,

$$F'(x) = \frac{1}{3}3x^2 = x^2$$

Are there any function F whose derivative is x^2? If we add any constant C to $F(x) = \frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f.

5.3. In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. It is then reasonable to ask if there are antiderivatives of a function f that cannot be obtained by adding some constant to a known antiderivative F. The answer is *no*.

Theorem 5.4. If $F'(x) = G'(x)$ on an interval I, then on I,

$$F(x) = G(x) + C$$

for some constant C.

Definition 5.5. The *collection* of all antiderivatives of a function $f(x)$ is called the **indefinite integral** and is denoted by

$$\int f(x)dx$$

If we know one function $F(x)$ for which $F'(x) = f(x)$, then

$$\int f(x)dx = F(x) + C$$

where C is any arbitrary constant. This constant is called the **constant of integration**. We refer the symbol $\int$ as the **integral sign** and $f(x)$ as the **integrand**.

- Graphs of antiderivatives of a function f are called **integral curves** of f.

5.6. Many basic integration formulas can be obtained directly from their companion differentiation formulas:

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

(b) $\int e^x dx = e^x + C$

(c) $\int \frac{1}{x} dx = \ln|x| + C$

(d) $\int \cos(x) dx = \sin(x) + C$

(e) $\int \sin(x) dx = -\cos(x) + C$

5.7. Properties of the Indefinite Integral:

(a) $\int kf(x)dx = k \int f(x)dx$

(b) $\int [f(x) \pm g(x)]dx = \int f(x) \pm \int g(x)dx$

(c) **Linearity of Indefinite Integral**:

$$\int af(x) + bg(x)dx = a \int f(x)dx + b \int g(x)dx$$

- These equations must be applied carefully to avoid errors and unnecessary complexities arising from the constants of integration.

Example 5.8. Evaluate the following integrals.

(a) $\int x^5 dx$
Solution:

(b) $\int \frac{1}{\sqrt{x}} dx$
Solution:

(c) $\int \pi \sqrt[5]{u^3} + 1 du$
Solution:

(d) $\int \frac{2}{x} + \frac{x^3}{\sqrt{2}} dx$
Solution:

5.9. Sometimes it is useful to rewrite an integrand in a different form before performing the integration.

Example 5.10. Evaluate the following integrals.

(a) $\int \frac{x+1}{x} dx$
Solution:

(b) $\int \sqrt{t}(t+1)dt$
Solution:

(c) $\int \frac{e^{-u}+1}{e^{-u}} du$
Solution:

5.11. Slope Field and Differential Equation: Finding antiderivatives of $f(x)$ is the same as finding a function $F(x)$ such that $y = F(x)$ satisfies the **differential equation**

$$\frac{dy}{dx} = f(x). \tag{13}$$

If we interpret dy/dx as the slope of a tangent line, then at a point (x, y) on an integral curve of Equation (13), the slope of the tangent line is $f(x)$. Note that these slopes can be obtained without actually solving the differential equation.

A geometric description of the integral curves of a differential equation (13) can be obtained by

(a) choosing a rectangular grid of points in the xy-plane,

(b) calculating the slopes of the tangent lines to the integral curves at the gridpoints, and

(c) drawing small portions of the tangent lines through those points.

The resulting picture, which is called a **slope field** or **direction field** for the equation, shows the "direction" of the integral curves at the grid points. With sufficiently many gridpoints it is often possible to visualize the integral curves themselves.

Example 5.12. Figure 5.2.3a shows a slope field for the differential equation $dy/dx = x^2$, and figure 5.2.3b shows that same field with the integral curves imposed on it.

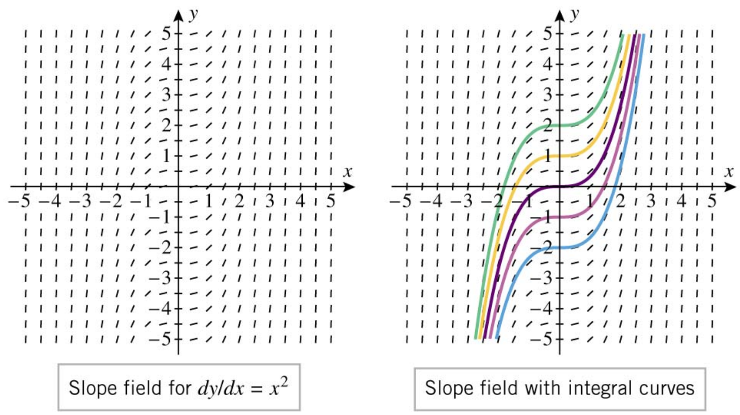

Figure 5.2.3 (*a*) (*b*)

5.2 Integration by Substitution

The substitution technique is based on reversing the chain rule. By chain rule we have

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x).$$

If F is an antiderivative of f, then we can write

$$\frac{d}{dx}\left[F(g(x))\right] = f(g(x))g'(x).$$

Therefore

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

5.13. Method of u-Substitution:

(a) Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \quad du = g'(x)dx$$

produces an integral that is expressed entirely in terms of u and du.

(b) Try to evaluate the resulting integral in terms of u.

(c) Replace u by $g(x)$.

Example 5.14. Evaluate the following integrals

(a) $\int 2x(x^2+1)^{99}dx$
Solution:

(b) $\int \frac{3x^2}{\sqrt{x^3+3}} dx$
Solution:

(c) $\int \frac{2 \ln x}{x} dx$
Solution:

(d) $\int \frac{x+1}{x^2+2x} dx$
Solution:

(e) $\int \frac{x+1}{x^2+2x} dx$
Solution:

(f) $\int x e^{x^2-1} dx$
Solution:

(g) $\int \cos^3 x \sin x dx$
Solution:

Example 5.15. Evaluate the following integrals

(a) $\int x^2 \sqrt{x-1} dx$
Solution:

(b) $\int \cos^3 x dx$
Solution:

5.3 Riemann sums

5.16. Numerical Approximations of Area

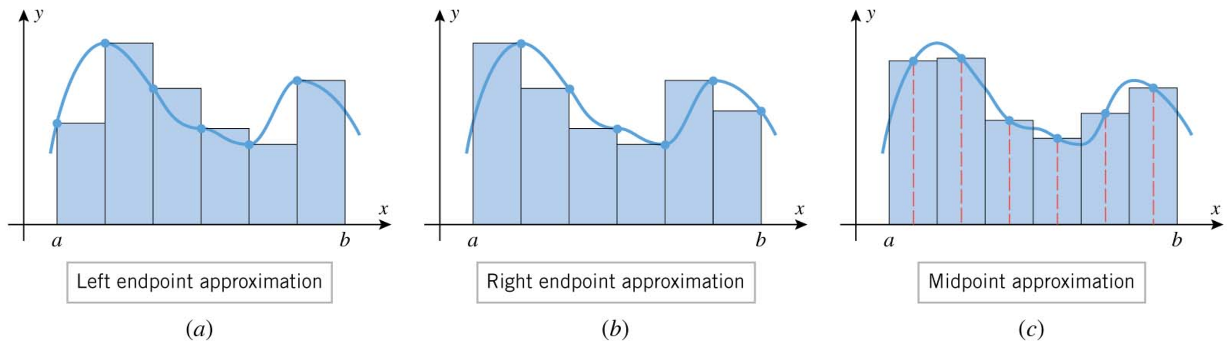

Figure 5.4.9

Example 5.17. Find the left endpoint approximation and the left endpoint approximation of the area under the curve $f(x) = x^2$ over the interval $[0, 1]$ with $n = 5$. Illustrate each part with a graph of f that includes the rectangles whose areas are represented in the sum.
Solution:

Example 5.18. Find the left endpoint approximation and the left endpoint approximation of the area under the curve $f(x) = \frac{1}{x}$ over the interval $[0, 1]$ with $n = 5$. Illustrate each part with a graph of f that includes the rectangles whose areas are represented in the sum.
Solution: