Sirindhorn International Institute of Technology
Thammasat University
Department of Common and Graduate Studies

MAS 116: Lecture Notes 5

Semester: 3/2008
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4 The Derivative in Graphing and Applications

We will study various applications of the derivative. First we will develop mathematical tools that can be used to determine the exact shape of a graph and the precise locations of its key features.

4.1 Increasing and Decreasing Functions

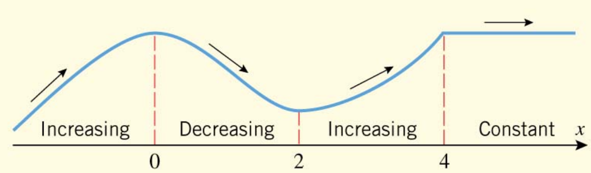


Figure 4.1.1

Definition 4.1. Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is (strictly) **increasing** (and denoted by $\nearrow$) on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

- The graph of the function rises while moving from left to right

(b) f is (strictly) **decreasing** (and denoted by $\searrow$) on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

- The graph of the function falls while moving left to right

(c) f is constant on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.

Theorem 4.2. Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

- Observe that the derivative conditions in Theorem 4.2 are only required to hold inside the interval $[a, b]$, even though the conclusions apply to the entire interval.

- Although stated for closed interval, Theorem 4.2 is applicable on any interval I on which f is continuous.

Example 4.3. Use the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ in Figure 4.1.6 to make a conjecture about the intervals on which f is increasing or decreasing.

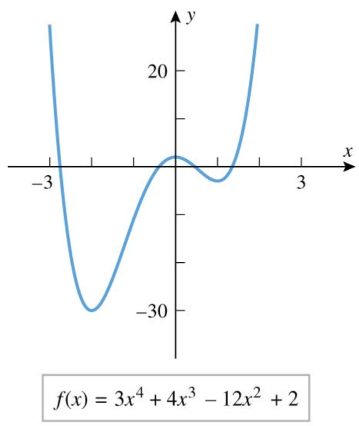


Figure 4.1.6

Now, use Theorem 4.2 to determine whether your conjecture is correct.
Solution:

Definition 4.4. A value $x = c$ is a **critical point** for a function $f(x)$ if c is in the domain of the function $f(x)$ and $f'(c) = 0$ or $f'(c)$ does not exist.

- To distinguish between the two types of critical points, we call x a **stationary point** of f if $f'(x) = 0$.

Example 4.5. In our Example 4.3 above, the point $x = -2, 0, 1$ are critical points since all of them have slope 0.

4.6. Procedure for finding where a function is increasing or decreasing:

(a) Find the critical points (look for when $f'(x) = 0$) or $f'(x)$ does not exist.

(b) Use the critical points to find the subintervals on which $f'(x)$ has constant sign.

(c) Select a convenient test value on each subinterval. Evaluate the value of f' at this number. The sign of $f'(x)$ at this test value will be the sign of each subinterval.

(d) If the subinterval has negative sign, then the function f is decreasing on this subinterval. If the subinterval has positive sign, then the function f is increasing on this subinterval.

4.2 Concavity

Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of *curvature*.

Definition 4.7. Intuitively, if the graph of a function bends upward ($\smile$), we say that the function is **concave up**. If the graph bends downward ($\frown$), we say that the function is **concave down**. A straight line is neither concave up or nor concave down.

Formally, if f is differentiable on an open interval I, then

(a) f is said to be **concave up** on I if f' is increasing on I, and

(b) f is said to be **concave down** on I if f' is decreasing on I.

Theorem 4.8 (Test for Concavity)**.** Let f be twice differentiable on an open interval $I = (a, b)$.

(a) If $f''(x) > 0$ for every value of x in I, then the graph of f is concave up ($\cup$) on I.

(b) If $f''(x) < 0$ for every value of x in I, then the graph of f is concave down ($\cap$) on I.

Example 4.9. Figure 41.10 shows the graph of the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down.

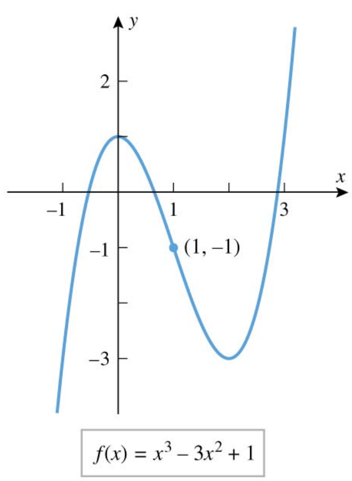


Figure 4.1.10

Solution:

Definition 4.10. If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an **inflection point at** x_0, and we call the point $(x_0, f(x_0)$ on the graph of f an **inflection point** of f.

- It is NOT always that case that the inflection points of f occurred whenever $f''(x) = 0$.
- If a function f has an inflection point at $x = x_0$ and $f''(x_0)$ exists, then $f''(x_0) = 0$.
- An inflection point may occur where $f''(x)$ is not defined.

Example 4.11. In our Example 4.9, there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. The inflection point is $(1, f(1)) = (1, -1)$.

Example 4.12. Find the inflection points, if any, of $f(x) = x^4$.
Solution:

4.3 Relative Extrema

If we imagine the graph of a function f to be a two dimensional mountain range with hills and valleys, then the tops of the hills are called "relative maxima," and the bottoms of the valleys are called "relative minima" (Figure 4.2.1).

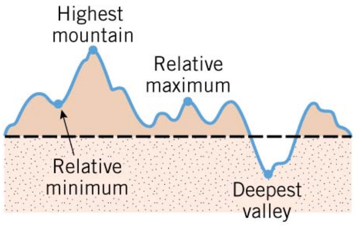


Figure 4.2.1

The relative maxima are the high points in their immediate vicinity, and the relative minima are the low points.

Definition 4.13.

(a) A function f is said to have a **relative maximum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is $f(x_0) \geq f(x)$ for all x in the interval.

(b) A function f is said to have a **relative minimum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is $f(x_0) \leq f(x)$ for all x in the interval.

(c) If f has either a relative maximum or a relative minimum at x_0, then f is said to have a **relative extremum** at x_0.

Theorem 4.14. Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f; that is, either $f'(x) = 0$ or f is not differentiable at x_0.

- It is NOT true that a relative extremum occurs at *every* critical point. See Figure 4.2.6.

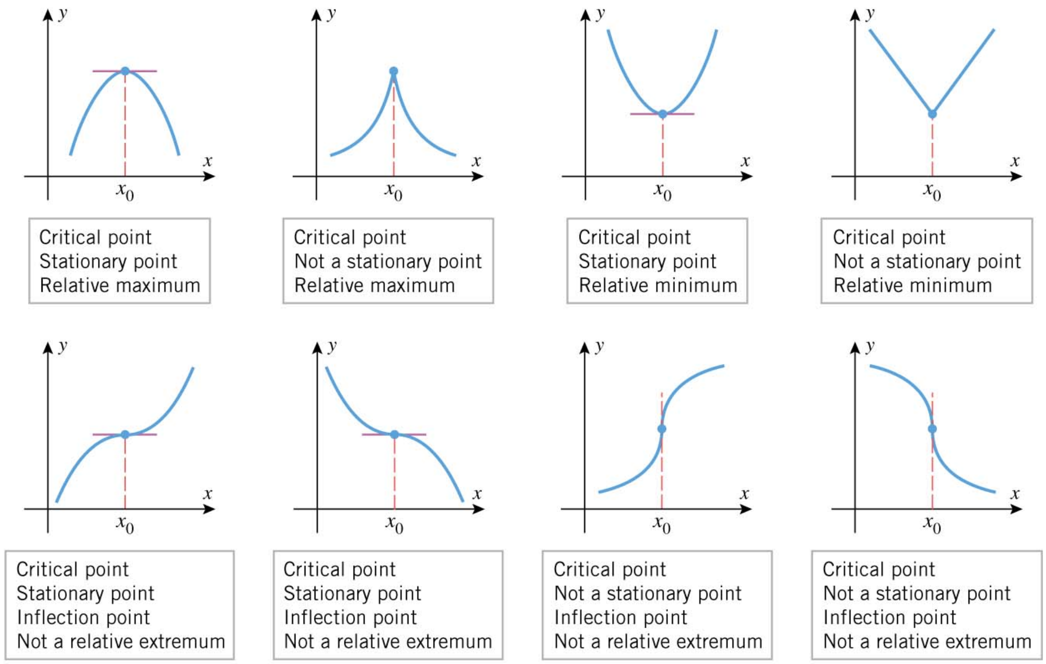


Figure 4.2.6

4.15. First Derivative Test: Suppose f is defined on (a, b) and c is a critical point in the interval (a, b).

(a) If $f'(x) > 0$ for x near and to the left of c and $f'(x) < 0$ for x near and to the right of c, then f has a relative maximum at c.

(b) If $f'(x) < 0$ for x near and to the left of c and $f'(x) > 0$ for x near and to the right of c, then $f(c)$ has a relative minimum at c.

(c) If the sign of $f'(x)$ is the same on both sides of c, then f does not have a relative extremum at x_0.

Example 4.16. Find all the relative extrema of the function $f(x) = x^3 - 3x^2 + 1$. Identify if they are relative maximum or relative minimum.
Solution:

4.17. Second Derivative Test: Suppose that f is twice differentiable at the point $x = c$. In addition, $f'(c) = 0$.

(a) If $f''(c) > 0$, then $f(c)$ is a relative minimum at $x = c$.

(b) If $f''(c) < 0$, then $f(c)$ is a relative maximum at $x = c$.

(c) If $f''(c) = 0$, then the test is inconclusive; that is f may have a relative maximum, a relative minimum, or neither at x_0.

Example 4.18. Find all the relative extrema of the function $f(x) = x^3 - 3x^2 + 1$. Identify if they are relative maximum or relative minimum.
Solution:

4.4 Absolute Extrema

Definition 4.19. Let I be an interval in the domain of a function f.

(a) We say that f has an **absolute maximum** at a point x_0 in I if $f(x_0) \geq f(x)$ for all x in I.

(b) We say that f has an **absolute minimum** at a point x_0 in I if $f(x_0) \leq f(x)$ for all x in I.

(c) We say that f has an **absolute extremum** at a point x_0 if it has either an absolute maximum or an absolute minimum at that point.

4.20. In general, there is no guarantee that a function will actually have an absolute maximum or minimum on a given interval (Figure 4.4.1).

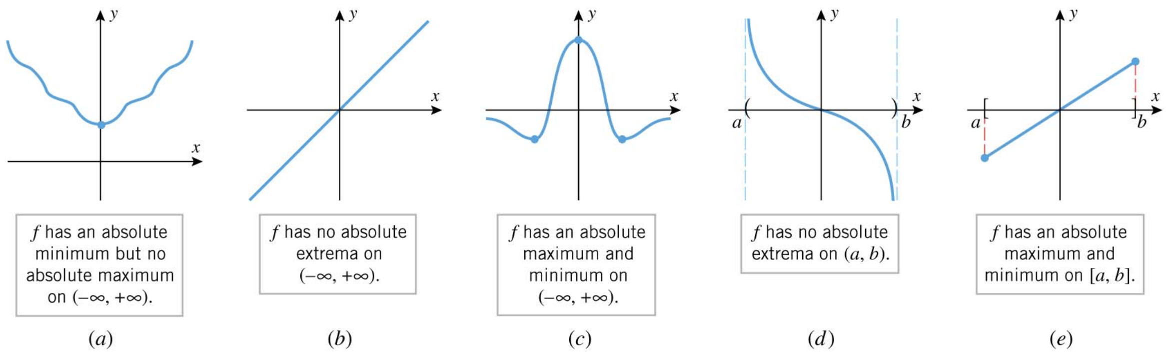


Figure 4.4.1

Theorem 4.21. If $f(x)$ is continuous on the closed and bounded interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

- If either the interval is not closed or f is not continuous on the interval, then f need not have absolute extrema on the interval.

Theorem 4.22. If f has an absolute extremum on an open interval (a, b), then it must occur at a critical point of f.

- This theorem is also valid on infinite open intervals.
- It follows from this theorem that if f is continuous on the finite closed interval $[a, b]$ then the absolute extrema occur either at the endpoints of the interval or at critical points inside the interval.

4.23. A Procedure for Finding the Absolute Extrema of a Continuous Function $f(x)$ on Finite Closed Interval $[a, b]$.

(a) Locate all critical points of f in (a, b).

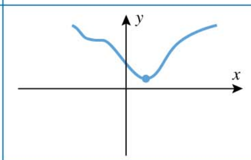
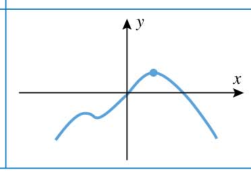
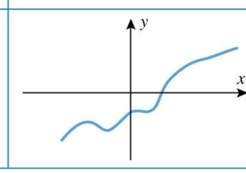
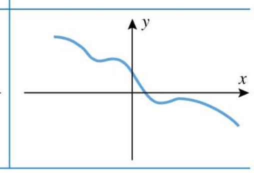
(b) Evaluate $f(x)$ at all the critical points and also at the two endpoints a and b.

(c) The absolute maximum of $f(x)$ on $[a, b]$ will be the largest number found in the previous step, whereas the absolute minimum of $f(x)$ on $[a, b]$ will be the smallest number found in the previous step.

4.24. Find the absolute maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 36x$ on the interval [1,5], and determine where these values occur.
Solution:

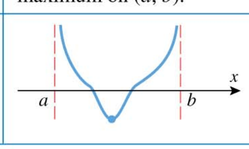
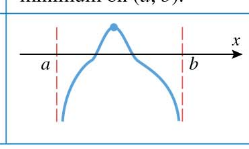
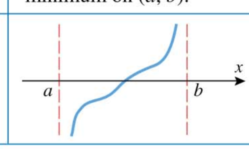
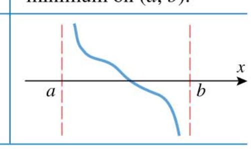
4.25. Absolute extrema on infinite intervals for continuous function:

Table 4.4.2

LIMITS	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = -\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
CONCLUSION IF *f* IS CONTINUOUS EVERYWHERE	f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.	f has an absolute maximum but no absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.
GRAPH				

4.26. Absolute extrema on open intervals for continuous function:

Table 4.4.3

LIMITS	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
CONCLUSION IF *f* IS CONTINUOUS ON (a, b)	f has an absolute minimum but no absolute maximum on (a, b).	f has an absolute maximum but no absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).
GRAPH				

Example 4.27. Determine whether the function

$$f(x) = \frac{1}{x^2 - x}$$

has any absolute extrema on the interval $(0, 1)$. If so, find them and state where they occur.
Solution:

4.5 Optimization: Applied Maximum and Minimum Problems

4.28. A Procedure for Solving Applied Maximum and Minimum Problems:

(a) Draw an appropriate figure an label the quantities relevant to the problem.

(b) Find a formula for the quantity to be maximized or minimized.

(c) Using the conditions stated in the problem to eliminate variables, express the quantity to be maximized or minimized as a function of one variable.

(d) Find the interval of possible values for this variable from the physical restrictions in the problem.

(e) If applicable, use the techniques of the preceding section to obtain the maximum or minimum.

Example 4.29. A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

Example 4.30. A farmer has 500 yards of fencing with which to fence in three sides of a rectangular pasture. A straight river will form the fourth side. Find the dimensions of the pasture of greatest area that the farmer can fence.

Example 4.31. What are the dimensions of an aluminum can that holds 8π in^3 of juice and that uses the least material (i.e., aluminum)? Assume that the can is cylindrical, and is capped on both ends.

Example 4.32. An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 4.5.3). What size should the squares be to obtain a box with the largest volume?

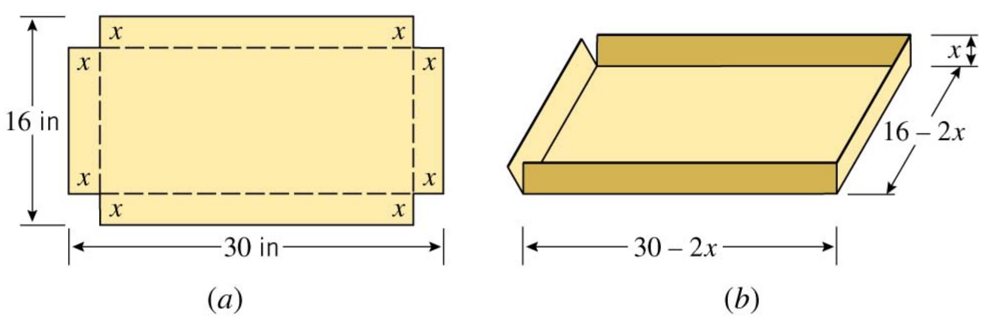
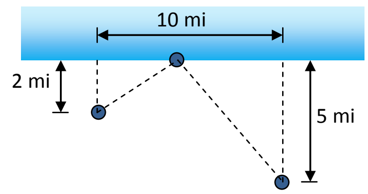


Figure 4.5.3

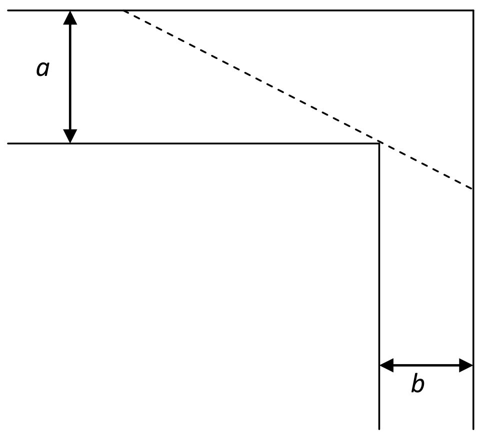
Example 4.33. A farmer wishes to enclose a rectangular field of area 450 square feet using and existing wall as one of the sides. The cost of the fence for the other three sides is \$3 per foot. Find the dimensions of the rectangular field that minimizes the cost of the fence.

Example 4.34. Of all rectangles with given area, A, which has the shortest diagonals?

Example 4.35. Two towns lie on the south side of a river. A pumping station is to be located to serve the two towns. A pipeline will be constructed from the pumping station to each of the towns along the line connecting the town and the pumping station. Locate the pumping station to minimize the amount of pipeline that must be constructed.



Example 4.36. Find the length of the longest ladder which can be carried around the corner of a corridor, whose dimensions are indicated in the figure below, if it is assumed that the ladder is carried parallel to the floor.



Example 4.37. A movie screen on a wall is 20 feet high and 10 feet above the floor. At what distance x from the front of the room should you position yourself so that the viewing angle of the movie screen is as large as possible?