Sirindhorn International Institute of Technology
Thammasat University
Department of Common and Graduate Studies

MAS 116: Lecture Notes 4

Semester: 3/2008
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3.10 Derivatives of Logarithmic Functions and Exponential Functions

3.67. For $x > 0$

$$\frac{d}{dx}\log_b(x) = \frac{d}{dx}\frac{\ln(x)}{\ln(b)} = \frac{1}{\ln b}\frac{d}{dx}\ln(x) = \frac{1}{x\ln b}. \tag{12}$$

3.68. If $b > 0$ and x is a real number,

$$b^x = e^{x\ln b}.$$

Therefore,

$$\frac{d}{dx}b^x = \frac{d}{dx}e^{x\ln b} = e^{x\ln b}\frac{d}{dx}(x\ln b) = b^x\ln b.$$

3.11 Derivatives of Inverse functions

In mathematics the term **inverse** is used to describe functions that reverse one another in the sense that each undoes the effect of the other. Inverse functions are very important in Mathematics as well as in many applied areas of science. The most famous pair of functions inverse to each other are the logarithmic and the exponential functions. Other functions like the tangent and arctangent play also a major role.

3.69. If f has an inverse, then the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.

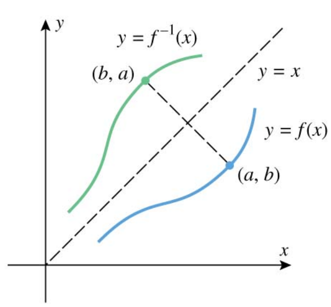

Figure 1.5.9

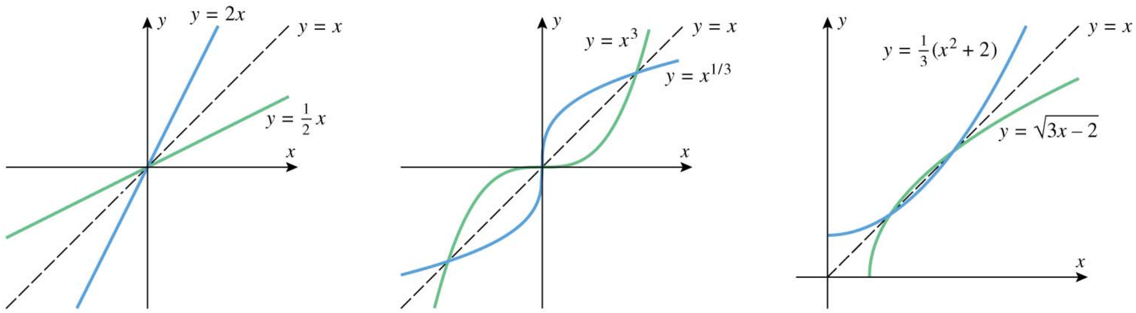

Figure 1.5.10

3.70. Geometrically, a function is differentiable at points where its graph has a *nonvertical tangent line*. Because the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$, it follows that the points where f^{-1} is not differentiable are reflections of the points where the graph of f has a *horizontal tangent line*. Stated algebraically, f^{-1} will fail to be differentiable at a point (b, a) on its graph if $f'(a) = 0$.

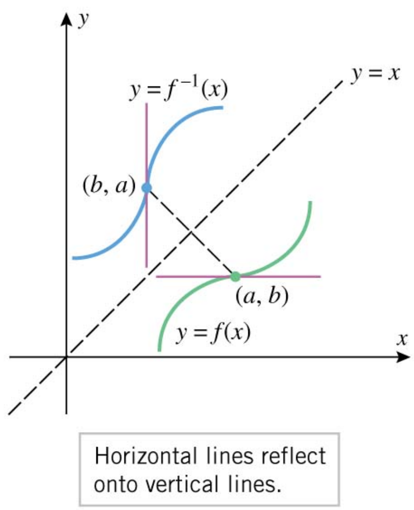

Figure 7.3.1

3.71. Assuming that f is differentiable at the point (a, b) and that $f'(a) \neq 0$, let us now try to find a relationship between the slope of the tangent line at the point (a, b) on the graph of f and the slope of the tangent line at the point (b, a) on the graph of f^{-1}.

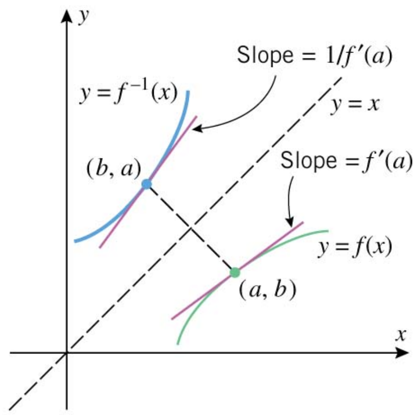

Figure 7.3.2

We know that the equation of the tangent line to the graph of f at the point (a, b) is

$$y - b = f'(a)(x - a).$$

An equation for the reflection of this line about the line $y = x$ can be obtained by interchanging x and y, so it follows that the tangent line to the graph of f^{-1} at the point (b, a) is

$$x - b = f'(a)(y - a)$$

which we can rewrite as

$$y - a = \frac{1}{f'(a)}(x - b).$$

- This equation tells us that the slope $(f^{-1})'(b)$ of the tangent line to the graph of f^{-1} at the point (b, a) is

$$(f^{-1})'(b) = \frac{1}{f'(a)} \text{ or equivalently } (f^{-1})'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}.$$

Theorem 3.72 (Differentiability of the Inverse Functions)**.** Suppose that the domain of a function f is an open interval I and that f is differentiable and one-to-one on this interval. Then f^{-1} is differentiable at any point x in the range of f at which $f'\left(f^{-1}(x)\right) \neq 0$, and its derivative is

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

Or if we let $y = f^{-1}(x)$, then we obtain the following alternative version of this formula:

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Example 3.73. Consider the function $f(x) = x^5 + 1$:

(a) Find $f^{-1}(x)$.
Solution:

(b) Find a formula for the derivative of $f^{-1}(x)$ from its expression found in the previous part.
Solution:

(c) Find a formula for the derivative of $f^{-1}(x)$ using the formula in Theorem 3.72.
Solution:

(d) Find a formula for the derivative of $f^{-1}(x)$ using implicit differentiation.
Solution:

Example 3.74. Find a formula for the derivative of b^x from (12).
Solution:

3.12 Logarithmic Differentiation

The method of logarithmic differentiation,in calculus, uses the properties of logarithmic functions (and implicit differentiation) to differentiate complicated functions that are composed of products, quotients, and powers.

Its easiest to see how this works in an example.

Example 3.75. Find dy/dx of the followings;

(a) $y = f(x) \times g(x)$

Solution:

$$\begin{aligned}
\ln y &= \ln\left(f\left(x\right)g\left(x\right)\right) = \ln f\left(x\right) + \ln g\left(x\right)\\
\frac{1}{y}\frac{dy}{dx} &= \frac{f'\left(x\right)}{f\left(x\right)} + \frac{g'\left(x\right)}{g\left(x\right)}\\
\frac{dy}{dx} &= y\left(\frac{f'\left(x\right)}{f\left(x\right)} + \frac{g'\left(x\right)}{g\left(x\right)}\right) = f\left(x\right)g\left(x\right)\left(\frac{f'\left(x\right)}{f\left(x\right)} + \frac{g'\left(x\right)}{g\left(x\right)}\right)\\
&= g\left(x\right)f'\left(x\right) + f\left(x\right)g'\left(x\right)
\end{aligned}$$

(b) $y = \frac{f(x)}{g(x)}$
Solution:

$$\begin{aligned}
\ln y &= \ln f\left(x\right) - \ln g\left(x\right)\\
\frac{1}{y}\frac{dy}{dx} &= \frac{f'\left(x\right)}{f\left(x\right)} - \frac{g'\left(x\right)}{g\left(x\right)}\\
\frac{dy}{dx} &= y\left(\frac{f'\left(x\right)}{f\left(x\right)} - \frac{g'\left(x\right)}{g\left(x\right)}\right) = \frac{f\left(x\right)}{g\left(x\right)}\left(\frac{f'\left(x\right)}{f\left(x\right)} - \frac{g'\left(x\right)}{g\left(x\right)}\right)\\
&= \frac{f'\left(x\right)}{g\left(x\right)} - \frac{f\left(x\right)g'\left(x\right)}{g^2\left(x\right)} = \frac{g\left(x\right)f'\left(x\right) - f\left(x\right)g'\left(x\right)}{g^2\left(x\right)}
\end{aligned}$$

(c) $y = x^{\sin x}$
Solution:

(d) $y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$
Solution:

3.13 Indeterminate Forms and L'Hôpital's Rule

An indeterminate form is an algebraic expression obtained in the context of limits. Recall from 2.27 that a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

in which $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an **indeterminate form of type 0/0**. We have seen previously that the limit can be obtained by factoring in the case of

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} x + 2 = 4.$$

We have also applied the squeezing theorem to find

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

However, there are many indeterminate forms for which algebraic manipulation will not directly produce the limit.

Definition 3.76. In general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x) \cdot g(x), \quad f(x)^{g(x)}, \quad f(x) \pm g(x)$$

is called an ***intermediate form*** if the limits of $f(x)$ and $g(x)$ individually exert **conflicting influences** on the limit of the entire expression. The indeterminate forms include $0^0, 0/0, 1^\infty, \infty - \infty, \infty/\infty, 0 \times \infty$, and ∞^0.

The indeterminate nature of a limit's form does not imply that the limit does not exist, as many of the examples above show. In many cases, algebraic elimination, L'Hôpital's rule, or other methods can be used to manipulate the expression so that the limit can be evaluated.

Theorem 3.77. (L'Hôpital's rule) Suppose f and g are differentiable function on an open interval containing $x = a$, except possibly at $x = a$, and that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \infty \text{ and } \lim_{x \to a} g(x) = \infty.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

- Note that in L'Hôpital's rule the numerator and denominator are differentiated individually. This is not the same as differentiating $f(x)/g(x)$.
- **Caution**: Applying L'Hôpital's ruleto limits that are not indeterminate forms can produce incorrect results.

Example 3.78. Evaluate

(a) $\lim_{x\to 2} \frac{x^2-4}{x-2}$
Solution:

(b) $\lim_{x\to\infty} \frac{\ln x}{x}$
Solution:

(c) $\lim_{x\to\infty} \frac{2x-1}{e^x}$
Solution:

(d) $\lim_{x\to 0} \frac{e^x-1}{x^3}$
Solution:

(e) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$
Solution:

LHospitals Rule works great on the two indeterminate forms 0/0 and ∞/∞. However, there are many more indeterminate forms out there as we saw earlier. Lets take a look at some of those and see how we deal with those kinds of indeterminate forms.

Example 3.79. Evaluate

(a) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$
Solution:

(b) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{e^x - 1}\right)$
Solution:

(c) $\lim_{x \to 0^+} x \ln x$
Solution:

(d) $\lim_{x \to \infty} \ln x - \ln(1 + x)$
Solution:

(e) $\lim_{x \to 0} (1+x)^{1/x}$

Solution: