Sirindhorn International Institute of Technology
Thammasat University
Department of Common and Graduate Studies

MAS 116: Lecture Notes 3

Semester: 3/2008
Instructors: Dr. Prapun Suksompong

3.18. Some History [2]:

- The tangent problem (the problem of trying to find an equation of the tangent line) has given rise to the branch of calculus called *differential calculus*, which was not invented until more than 2000 years after *integral calculus*.
- For the branch of calculus called *integral calculus*, the central problem is the area problem whose origins go back to the ancient Greeks.

3.3 Derivatives of Simple Functions

3.19. Derivative of a Constant: The derivative of a constant function is 0; that is, if c is any real number, then

$$\frac{d}{dx}c = 0$$

Example 3.20. Find the derivative of the following functions

(a) $f(x) = 13$
Solution:

(b) $f(x) = \sqrt{2}$
Solution:

(c) $f(x) = e + \pi$
Solution:

3.21. The Power Rule: If n is any real number (n may or may not be an integer),

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 3.22. Find the derivative of the following functions

(a) $f(x) = x^3$
Solution:

(b) $f(x) = \frac{1}{x^2}$
Solution:

(c) $f(x) = \sqrt{x}$
Solution:

3.23. Derivative of e^x: The derivative of e^x is again e^x; that is,

$$\frac{d}{dx}(e^x) = e^x.$$

3.24. Derivative of $\ln x$: If $x > 0$, then

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

3.4 Techniques of Differentiation

3.25. Constant Multiple Rule: If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x) = cf'(x).$$

In words, a constant factor can be moved through a derivative sign.

3.26. Sum and Difference Rules: If f and g are differentiable at x, then so are $f + g$ and $f - g$ and

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)] = f'(x) \pm g'(x).$$

In words, the derivative of a sum equals the sum of the derivatives, and the derivative of a difference equals the difference of the derivatives.

- Although the above sum and difference rules are stated for sums and differences of two functions, they can be extended to any finite number of functions.

Example 3.27. Find the derivative of the following functions

(a) $f(x) = \pi x + 2\ln x - 3\ln(5x)$
Solution:

(b) $f(x) = \frac{x+3}{\sqrt{x}}$
Solution:

(c) $f(x) = \sqrt[3]{x} - \frac{1}{\sqrt[3]{x}}$
Solution:

(d) $f(x) = \frac{(x+1)^2}{x}$
Solution:

3.28. Linearity of differentiation follows from the sum rule and the constant multiple rule. Let f and g be functions, with α and β fixed. Then,

$$\frac{d}{dx}(\alpha \cdot f(x) + \beta \cdot g(x)) = \alpha \frac{d}{dx} f(x) + \beta \frac{d}{dx} g(x),$$

or

$$(\alpha \cdot f + \beta \cdot g)' = \alpha \cdot f' + \beta \cdot g'.$$

3.29. Product Rule: If f and g are differentiable at x, then so is the product $f \cdot g$, and

$$\frac{d}{dx}(f(x) \cdot g(x)) = g(x) \cdot \frac{d}{dx} f(x) + f(x) \cdot \frac{d}{dx} g(x) = g(x)f'(x) + f(x)g'(x)$$

- If we let $u = f(x)$ and $v = g(x)$, we have

$$(uv)' = u'v + v'u$$

- In general, the derivative of a product of two functions is NOT the product of their derivatives.

Example 3.30. Find the derivative of the following functions

(a) $h(x) = e^x \ln x$
Solution:

(b) $h(x) = (2x - 3\ln x)(x + \frac{1}{\sqrt{x}})$
Solution:

3.31. Quotient Rule: If f and g are differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

- If we let $u = f(x)$ and $v = g(x)$, we have

$$\Big(\frac{u}{v}\Big)' = \frac{u'v - v'u}{v^2}$$

Example 3.32. Find the derivative of the following functions

(a) $h(x) = \frac{e^x}{x^2+1}$
Solution:

(b) $h(x) = \frac{\sqrt[4]{x}}{x^2-2x+1}$
Solution:

3.5 The Chain Rule

We want to find a formula for the derivative of $(f \circ g)(x) = f[g(x)]$ in terms of the derivatives of $f(x)$ and $g(x)$.

Theorem 3.33. If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or

$$\frac{d}{dx} f[g(x)] = f'[g(x)] \cdot g'(x)$$

- A convenient way to remember this formula is to call f the "outside function" and g the "inside function" in the composition $f(g(x))$. In which case, the derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

3.34. Generalized Derivative Formula: If we write $u = g(x)$, then we can rewrite the chain rule as

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}.$$

Example 3.35. Find the derivative of the following functions

(a) $f(x) = (x^2 + 3)^{78}$.
Solution:

(b) $f(x) = \frac{1}{(x^3+x-5)^4}$.
Solution:

(c) $f(x) = \sqrt{\sqrt{x} + 1}$.
Solution:

(d) $f(x) = \sqrt{x}(4x+3)^3$.
Solution:

(e) $f(x) = e^{x^3+4x^2+x-5}$.
Solution:

(f) $f(x) = \ln|x|$.
Solution:

(g) $f(x) = \ln\left|\frac{x+1}{x+2}\right|$.
Solution:

Example 3.36. Given that $f'(x) = \sqrt{3x+4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.
Solution:

Example 3.37. Find $f'(x^2 - 1)$ if

$$\frac{d}{dx}[f(x^2-1)] = x^3$$

Solution:

3.6 Higher Derivatives

3.38. The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is call the **second derivative** of f.

As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of f. These successive derivatives are denoted by

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad \dots$$

- If $y = f(x)$, then successive derivatives can also be denoted by

$$y', \quad y'', \quad y''', \quad \dots$$

- Another common notation is

$$\frac{d}{dx}[f(x)], \quad \frac{d^2}{dx^2}[f(x)], \quad \frac{d^3}{dx^3}[f(x)], \quad \dots$$

- These are called, in succession, the first derivative, the second derivative, the third derivative, and so forth.

Definition 3.39. The number of times that f is differentiated is called the order of the derivative.

Definition 3.40. A general nth order derivative can be denoted by

$$\frac{d^n y}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

and the value of a general nth order derivative at a specific point $x = x_0$ can be denoted by

$$\left.\frac{d^n y}{dx^n}\right|_{x=x_0} = f^{(n)}(x_0) = \left.\frac{d^n}{dx^n}[f(x)]\right|_{x=x_0}.$$

Example 3.41. Find the first and second derivative of the following

(a) $f(x) = 2x^3 + 2x^2 - 1$
Solution:

(b) $f(x) = e^x + 2\ln x$
Solution:

(c) $f(x) = \frac{x+1}{x}$
Solution:

3.7 Trigonometry Review

3.42. The **trigonometric functions** are functions of an angle. In modern usage, there are six basic trigonometric functions. For a positive acute angle , they can be defined as ratios of the sides of a right triangle.

$$\sin\theta = \frac{\text{side opposite }\theta}{\text{hypotenuse}} = \frac{y}{r}, \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite }\theta} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{side adjacent to }\theta}{\text{hypotenuse}} = \frac{x}{r}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent to }\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{side opposite }\theta}{\text{side adjacent to }\theta} = \frac{y}{x}, \qquad \cot\theta = \frac{\text{side adjacent to }\theta}{\text{side opposite }\theta} = \frac{x}{y}$$

3.43. All six trigonometric functions are listed below along with equations relating them to one another.

(a) Sine: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.

(b) Cosine: $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

(c) Tangent: $\tan\theta = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta} = \cot\left(\frac{\pi}{2} - \theta\right)$.

(d) Cotangent: $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \tan\left(\frac{\pi}{2} - \theta\right)$.

(e) Secant: $\sec\theta = \frac{1}{\cos\theta} = \csc\left(\frac{\pi}{2} - \theta\right)$.

(f) Cosecant: $\csc\theta = \frac{1}{\sin\theta} = \sec\left(\frac{\pi}{2} - \theta\right)$.

Example 3.44. All of the trigonometric functions of a positive acute angle θ can be constructed geometrically in terms of a unit circle.

3.45. It is only in special cases (some of which are provided in Table 3 below) that exact values for trigonometric functions can be obtained; usually a calculating utility or a computer program will be required.

Table 3

	$\theta = 0$ (0°)	$\pi/6$ (30°)	$\pi/4$ (45°)	$\pi/3$ (60°)	$\pi/2$ (90°)	$2\pi/3$ (120°)	$3\pi/4$ (135°)	$5\pi/6$ (150°)	π (180°)	$3\pi/2$ (270°)	2π (360°)
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1/2	$-1/\sqrt{2}$	$-\sqrt{3}/2$	–1	0	1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	—	$-\sqrt{3}$	–1	$-1/\sqrt{3}$	0	—	0
$\csc\theta$	—	2	$\sqrt{2}$	$2/\sqrt{3}$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	—	–1	—
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	—	–2	$-\sqrt{2}$	$-2/\sqrt{3}$	–1	—	1
$\cot\theta$	—	$\sqrt{3}$	1	$1/\sqrt{3}$	0	$-1/\sqrt{3}$	–1	$-\sqrt{3}$	—	0	—

3.46. Difference identity:

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{7}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right) \tag{8}$$

Example 3.47. Use Example 3.44 to show that

$$\sin x \le x \le \tan x. \tag{9}$$

for $0 < x < \frac{\pi}{2}$.

Using the fact that $\sin x > 0$ for $0 < x < \frac{\pi}{2}$, we obtain

$$\cos x \le \frac{\sin x}{x} \le 1. \tag{10}$$

The inequality (10) also holds for $-\frac{\pi}{2} < x < 0$. To see this, replace x ny $-x$ and use the identities $\sin(-x) = -\sin(x)$, and $\cos(-x) = \cos x$.

Example 3.48. Use (10) and the squeezing theorem to show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.
Solution:

Example 3.49. Use Example 3.48 to show that

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0.$$

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x}\frac{1}{1+\cos x}\\ &= \lim_{x\to 0}\frac{\sin^2 x}{x}\frac{1}{1+\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\frac{\sin x}{1+\cos x}\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right)=(1)\left(\frac{0}{1+1}\right)=0\end{aligned}$$

Example 3.50. Use Example 3.48 to show that

$$\lim_{x\to 0}\frac{\tan x}{x}=1$$

Solution:

$$\lim_{x\to 0}\frac{\tan x}{x}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right)=(1)\,(1)=0.$$

Example 3.51. Some more interesting limits:

(a) $\lim\limits_{x\to 0}\frac{\sin\beta x}{x}=\beta$.

(b) $\lim\limits_{x\to 0}\frac{\sin\beta_1 x}{\sin\beta_2 x}=\frac{\beta_1}{\beta_2}$.

3.8 Derivatives of Trigonometric Functions

We will assume that the variable x in the trigonometric functions $\sin x, \cos x, \tan x, \cot x, \sec x$, and $\csc x$ is measured in radians.

Example 3.52. Show that $\frac{d}{dx}\sin x=\cos x$.
Solution: Recall the difference identity (7)

$$\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right).$$

Hence,

$$\frac{\sin(x+h)-\sin x}{h}=2\frac{\sin\left(\frac{h}{2}\right)}{h}\cos\left(x+\frac{h}{2}\right)=\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\cos\left(x+\frac{h}{2}\right)$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\sin x=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} &= \left(\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\right)\\ &= (1)\,(\cos x)=\cos x.\end{aligned}$$

Example 3.53. $\frac{d}{dx}\csc x=\frac{d}{dx}\frac{1}{\sin x}=-\frac{1}{\sin^2 x}\cos x=-\csc x\cot x$

Example 3.54. $\frac{d}{dx}\cos x=\frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right)=-\cos\left(\frac{\pi}{2}-x\right)=-\sin x$

Example 3.55. Show that $\frac{d}{dx}\tan x = \sec^2 x$.

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} &= \frac{\cos x \frac{d}{dx}\sin x - \left(\frac{d}{dx}\cos x\right)\sin x}{\cos^2 x} \\ &= \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Example 3.56. Consider two function $f(x)$ and $g(x)$ with the following relation:

$$g(x) = f\left(\frac{\pi}{2} - x\right). \tag{11}$$

Then, by chain rule,

$$g'(x) = -f'\left(\frac{\pi}{2} - x\right).$$

If the formula for $f'(x)$ consists only of terms involving trigonometric functions of x, then $f'\left(\frac{\pi}{2} - x\right)$ is simply $f'(x)$ with all of its trigonometric function replaced by their "cofunction". In fact, we have already follow this procedure in Example 3.54. Therefore, by Example 3.53 and 3.55, we have

$$\frac{d}{dx}\cot x = -\csc^2 x$$

and

$$\frac{d}{dx}\sec x = +\sec x \tan x.$$

3.57. The examples above find the derivative of the trigonometric functions which we summarize here:

$$\begin{aligned}\frac{d}{dx}[\sin(x)] &= \cos(x) \\ \frac{d}{dx}[\cos(x)] &= -\sin(x) \\ \frac{d}{dx}[\tan(x)] &= \sec^2(x) \\ \frac{d}{dx}[\sec(x)] &= \sec(x)\tan(x) \\ \frac{d}{dx}[\cot(x)] &= -\csc^2(x) \\ \frac{d}{dx}[\csc(x)] &= -\csc(x)\cot(x)\end{aligned}$$

Example 3.58. Find the derivative of the following functions

(a) $f(x) = \frac{5}{x^2} + \sin(x)$
Solution:

(b) $f(x) = (x^2 + 1)\cos x$
Solution:

(c) $f(x) = \frac{\sec(x)}{1+\tan(x)}$
Solution:

Example 3.59. Find $f''(\pi/4)$ if $f(x) = \sec(x)$
Solution:

Example 3.60. Find the equation of the line tangent to the graph of $\cos x$ at $x = 0, \pi/2, \pi$
Solution:

3.9 Implicit Differentiation

An equation of the form $y = f(x)$ is said to define y **explicitly** as a function of x because the variable y appears alone on one side of the equation. However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$\sin(x\,y) = y$$

is not of the form $y = f(x)$.

Definition 3.61. We will say that a given equation in x and y defines the function f implicitly if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.

3.62. In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. We can differentiate both sides of the equation and then solve for dy/dx, treating y as a (temporarily unspecified) differentiable function of x.

- The resulting formula may involve both x and y. In order to obtain a formula for dy/dx that involves x alone, we would have to solve the original equation for y in terms of x and then substitute into the formula we already have. Sometimes, it is impossible to solve for y in terms of x and we are forced to leave the formula for dy/dx in terms of x and y.

3.63. Implicit differentiation can be used to find formula for derivatives of rational powers of x from formula for derivatives of integer powers of x. The idea is to write $y = x^{m/n}$ as $y^n = x^m$. (See p. 202–203 in [1].)

Example 3.64. Let $7 = x^3 + xy + y^2$. Find $\frac{dy}{dx}$.
Solution:

Example 3.65. Let $\sqrt{y} - \sin x = e^x$. Find $\frac{dy}{dx}$.
Solution:

Example 3.66. Let $x^4 - 2xy^3 + y^5 = 32$. Find $\frac{dy}{dx}$ and also the slope of the tangent line to the curve at $(0, 2)$.
Solution: