Name ________________________ ID3 _ _ _

EES 351: Principles of Communications **2020/1**

HW 7 — Due: October 28, 11:59 PM

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 5 pages.

(b) Some MATLAB scripts are available from `http://www2.siit.tu.ac.th/prapun/ees351/EES351_2020_HW_4_MATLAB.zip`.

(c) (1 pt) Two choices for submission:

(i) Online submission via Google Classroom

- PDF only. Paper size should be the same as the posted file.
- Only for those who can directly work on the posted PDF file using devices with pen input.
- No scanned work, photos, or screen capture.
- Your file name should start with your 10-digit student ID: "5565242231 351 HW4.pdf"

(ii) Hardcopy submission: Work and write your answers **directly on a hardcopy of the posted file** (not on another blank sheet of paper).

(d) (1 pt) Write your first name and the last three digits of your student ID in the spaces provided on the upper-right corner of this page.

(e) (8 pt) Try to solve all problems.

(f) Late submission will be heavily penalized.

Problem 1. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 6.1. Note that V and T_b are some positive constants. Your answers should be given in terms of them.

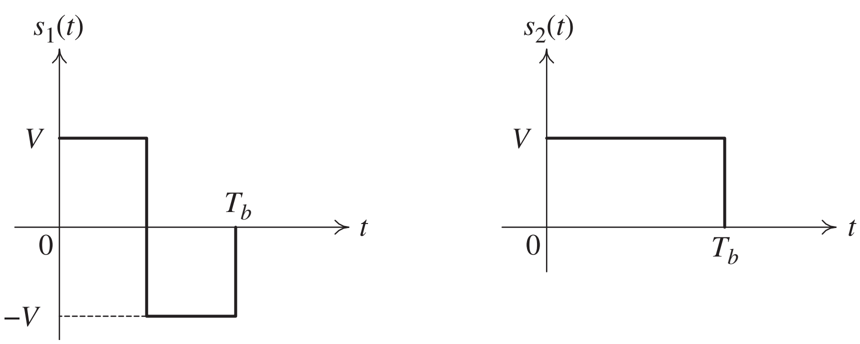


Figure 7.1: Signal set for Question 1

(a) Find the energy in each signal.

$$\left.\begin{aligned} E_{s_1} &= \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b. \\ E_{s_2} &= \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b. \end{aligned}\right\} \text{same (total) energy}$$

(b) Are they energy signals?

Because V and T_b are positive constants, we know that $V^2 T_b$ is positive and finite. Therefore, $0 < E_{s_1}, E_{s_2} < \infty$. Hence, both s_1 and s_2 are energy signals $\Rightarrow$ Yes

(c) Are they power signals?

No. Because they are energy signals, they can not be power signals.

(d) Find the (average) power in each signal.

All energy signals have 0 (average) power.

[See the comment 1 at the end]

(e) Are the two signals $s_1(t)$ and $s_2(t)$ orthogonal? (Two signals are orthogonal if their inner product is 0.)

$$\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t)\, s_2(t)\, dt = \int_0^{T_b/2} V \times V\, dt + \int_{T_b/2}^{T_b} (-V)\times(V)\, dt = V^2 \frac{T_b}{2} - V^2 \frac{T_b}{2} = 0.$$

Because $\langle s_1, s_2 \rangle = 0$, we know that s_1 and s_2 are orthogonal.

Problem 2. (Power Calculation) For each of the following signals $g(t)$, find (i) its corresponding power $P_g = \langle |g(t)|^2 \rangle$, (ii) the power $P_x = \langle |x(t)|^2 \rangle$ of $x(t) = g(t)\cos(10t)$, and (iii) the power $P_y = \langle |y(t)|^2 \rangle$ of $y(t) = g(t)\cos(50t)$

(a) $g(t) = 3\cos(10t + 30°)$.

Assume $f_0 \neq 0$ $\quad$ $A = 3$.

(a.i) $g(t) = A\cos(2\pi f_0 t + \theta) \Rightarrow P_g = \frac{|A|^2}{2} \Rightarrow P_g = \frac{|3|^2}{2} = \frac{9}{2} = 4.5.$

(a.ii) $x(t) = g(t)\cos(10t) = (3\cos(10t + 30°))(\cos(10t)) \overset{\text{product-to-sum formula}}{=} \frac{3}{2}\left(\cos(20t + 30°) + \underbrace{\cos(30°)}_{\text{a constant}}\right)$

non overlapping in the freq. domain

[See comment 3 at the end]

$$P_x = \left(\frac{3}{2}\right)^2 \left(\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2\right) = \frac{9}{4}\left(\frac{1}{2} + \frac{3}{4}\right) = \frac{9}{4}\left(\frac{5}{4}\right) = \frac{45}{16} \approx 2.813$$

[See comment 2 and comment 4 at the end]

(a.iii) $y(t) = g(t)\cos(50t) = (3\cos(10t + 30°))(\cos(50t)) \overset{\text{product-to-sum formula}}{=} \frac{3}{2}\left(\cos(60t + 30°) + \cos(40t - 30°)\right)$

different freq.

$$P_y = \left(\frac{3}{2}\right)^2\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{9}{4} \times 1 = \frac{9}{4}$$

[See comment 5 at the end]

[See comment 6 at the end for alternative solution]

(b) $g(t) = 3\cos(10t + 30°) + 4\cos(10t + 120°)$. (Hint: First, use phasor form to combine the two components into one sinusoid.)

(b.i) $g(t) = 3\cos(10t + 30°) + 4\cos(10t + 120°) = \mathrm{Re}\left\{ \underbrace{(3\angle 30° + 4\angle 120°)}_{\approx 0.5981 + 4.9641j \approx 5\angle 83.13°} e^{j10t} \right\}$

$= 5\cos(10t + 83.13°)$

Note that we do not need the phase 83.13° to calculate the average power. Also, we can get the magnitude "5" simply by noticing the 90° difference between $3\angle 30°$ and $4\angle 120°$.

$P_g = 5^2 \times \frac{1}{2} = \frac{25}{2} = 12.5$

(b.ii) $x(t) = g(t)\cos(10t) = 5\cos(10t + 83.13°)\cos(10t)$

$= \frac{5}{2}\left(\cos(20t + 83.13°) + \cos(83.13°)\right)$

$P_x = \left(\frac{5}{2}\right)^2\left(\frac{1}{2} + \cos^2(83.13°)\right) = \frac{25}{8}\left(1 + 2\cos^2(83.13°)\right) \approx 3.214$

(b.iii) Note that $G(f)$ is still at $\pm\frac{10}{2\pi}$ as in part (a.iii).

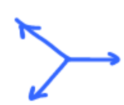
Therefore, $G(f - \frac{50}{2\pi})$ and $G(f + \frac{50}{2\pi})$ still do not overlap in the freq. domain.

$P_y = \frac{1}{2}P_g = \frac{25}{4} = 6.25$

(c) $g(t) = 3\cos(10t) + 3\cos(10t + 120°) + 3\cos(10t + 240°)$

(c.i) Look at the three components of $g(t)$ in their phasor representation.

We have $3\angle 0° + 3\angle 120° + 3\angle 240° = 0$



clear when you draw the three vectors

Therefore, $g(t) = 0$. Hence, $P_g = 0$.

(c.ii) $x(t) = 0 \Rightarrow P_x = 0$

(c.iii) $y(t) = 0 \Rightarrow P_y = 0$

Extra Question

Here is an optional question for those who want more practice.

Problem 3. Consider a signal $g(t)$. Recall that $|G(f)|^2$ is called the **energy spectral density** of $g(t)$. Integrating the energy spectral density over all frequency gives the signal's total energy. Furthermore, the energy contained in the frequency band I can be found from the integral $\int_I |G(f)|^2 df$ where the integration is over the frequencies in band I. In particular, if the band is simply an interval of frequency from f_1 to f_2, then the energy contained in this band is given by

$$\int_{f_1}^{f_2} |G(f)|^2 df. \tag{7.1}$$

In this problem, assume

$$g(t) = 1[-1 \le t \le 1].$$

(a) Find the (total) energy of $g(t)$.

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} \left(1[-1 \le t \le 1]\right)^2 dt = \int_{-1}^{1} 1\, dt = 2.$$

Remark: We can also try to find E_g from the freq. domain.

In part (b), we will show that $G(f) = 2\,\mathrm{sinc}(2\pi f)$.

Therefore,

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt \underset{\text{Parseval's theorem}}{=} \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} \left(2\,\mathrm{sinc}(2\pi f)\right)^2 df$$

$\mu = 2\pi f$, $d\mu = 2\pi\, df$

$$= \frac{1}{2\pi} \times 4 \int_{-\infty}^{\infty} \mathrm{sinc}^2(\mu)\, d\mu \overset{\text{Ex. 2.44.6}}{=} \frac{1}{2\pi} \times 4 \times \pi = 2$$

(b) Figure 6.2 define the main lobe of a sinc pulse. It is well-known that the main lobe of the sinc function contains about 90% of its total energy. Check this fact by first computing the energy contained in the frequency band occupied by the main lobe and then compare with your answer from part (a).

Hint: Find the zeros of the main lope. This give f_1 and f_2. Now, we can apply (6.1). `MATLAB` or similar tools can then be used to numerically evaluate the integral.

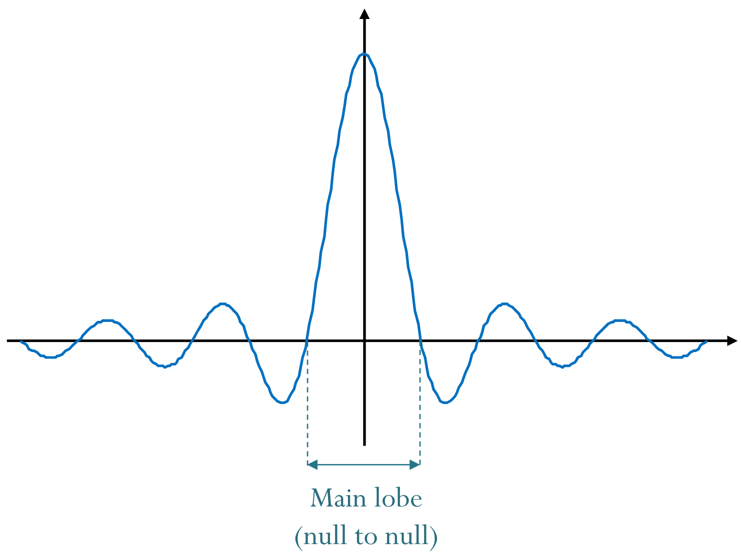
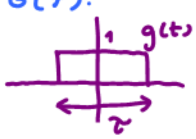


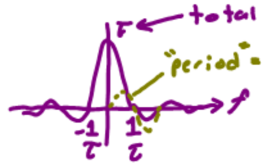
Figure 7.2: Main lobe of a sinc pulse

First, we need $G(f)$.

Recall that



$\xrightarrow{\mathcal{F}}$



← total area in time-domain

"period" = $\frac{2}{\tau}$ ⇒ "freq" = $\frac{\tau}{2}$ → $\sin(2\pi \frac{\tau}{2} f)$

$G(f) = \tau \operatorname{sinc}(\pi f \tau)$

Here, $\tau = 2$. So, $G(f) = 2\operatorname{sinc}(2\pi f)$

The main lobe occupies an interval of frequency from $f_1 = -\frac{1}{\tau} = -\frac{1}{2}$ to $f_2 = +\frac{1}{\tau} = +\frac{1}{2}$.

So the energy contained in the band $B = [f_1, f_2]$ is given by $\int_{-1/2}^{1/2} (2\operatorname{sinc}(2\pi f))^2 \, df \approx 1.8056$ ← MATLAB

Compared with the answer from part (a), this is $\approx 90\%$ of the total energy.

(c) Suppose we want to include more energy by considering wider frequency band. Let this band be the interval $I = [-f_0, f_0]$. Find the minimum value of f_0 that allows the band to capture at least 99% of the total energy in $g(t)$.

Using MATLAB, we can look at the fraction of energy as a function of f_0.

We found that at around $f_0 \approx 5.1$, the fraction begins to exceed 99%.

Comment 1:

All energy signals have 0 average

Consider $P_g = \lim_{T\to\infty} \int_{-T/2}^{T/2} |g(t)|^2 dt$.

Note that $|g(t)|^2$ is always nonnegative. Therefore,

$$0 \le \int_{-T/2}^{T/2} |g(t)|^2 dt \le \int_{-\infty}^{\infty} |g(t)|^2 dt = E_g$$

g(t) is an energy signal; so this is a finite number.

$$0 \le \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt \le \frac{1}{T} E_g$$

$$\Big\downarrow T\to\infty \qquad \Big\downarrow T\to\infty$$

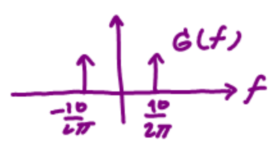
Here, we take the limit as $T \to \infty$.

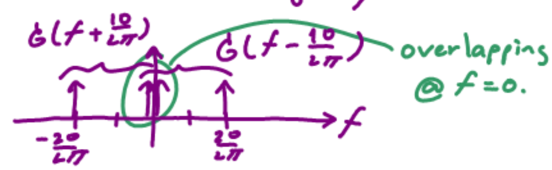
$$0 \le P_g \le 0$$

Therefore, $P_g = 0$.

Comment 2:

Note that although $x(t) = g(t)\cos(2\pi f_0 t)$, we can't use $P_x = \frac{1}{2}P_g$ because $G(f-f_0)$ and $G(f+f_0)$ overlap in the frequency domain.





Comment 3: A property that we frequently use in power calculation

Let $v(t) = a\,u(t)$. Then

$$P_v = \langle |v^2(t)| \rangle = \langle |a^2 u^2(t)| \rangle = |a|^2 \langle |u^2(t)| \rangle = |a|^2 P_u$$

Comment 4:

In general, for $x(t) = a\cos(2\pi f_0 t + \theta)\cos(2\pi f_0 t + \phi)$,

applying the product-to-sum formula gives

$$x(t) = \frac{a}{2}\Big(\cos\big(2\pi(2f_0)t + \theta + \phi\big) + \cos(\theta - \phi)\Big)$$

When $f_0 \ne 0$, the two cosine components do not overlap in the frequency domain. Hence, the power of their sum is the same as the sum of their power.

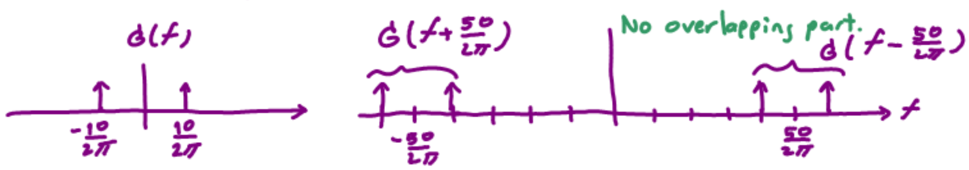
Therefore, $P_x = \left|\frac{a}{2}\right|^2 \left(\frac{1}{2} + \cos^2(\theta - \phi)\right)$.

Here, $a = 3$, $\theta = 0$, $\phi = 30°$.

Therefore, $P_x = \left(\frac{3}{2}\right)^2\left(\frac{1}{2} + \cos^2(30°)\right) = \frac{9}{4}\left(\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2\right) = \frac{9}{4}\left(\frac{1}{2} + \frac{3}{4}\right) = \frac{45}{16}$

Comment 5:

Note that $P_y = \frac{1}{2}P_g$ because $G(f - \frac{50}{2\pi})$ and $G(f + \frac{50}{2\pi})$ do not overlap.



Comment 6:

(a. i) $g(t) = 3\cos(10t + 30^\circ) = \frac{3}{2}\left(e^{j(10t+30^\circ)} + e^{-j(10t+30^\circ)}\right)$

$= \frac{3}{2} e^{j30^\circ} e^{j10t} + \frac{3}{2} e^{-j30^\circ} e^{-j10t}$

$P_g = \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 2 \times \frac{9}{4} = \frac{9}{2} = 4.5$

(a. ii) $y(t) = g(t)\cos(50t) = \frac{3}{2}\left(e^{j30^\circ}e^{j10t} + e^{-j30^\circ}e^{-j10t}\right)\frac{1}{2}\left(e^{j50t} + e^{-j50t}\right)$

$= \frac{3}{4}\left(e^{j30^\circ}e^{j60t} + e^{-j30^\circ}e^{j40t} + e^{j30^\circ}e^{-j40t} + e^{-j30^\circ}e^{-j60t}\right)$

All of the complex exponential functions have ditinct frequencies.

$P_y = \left(\frac{3}{4}\right)^2 \left(1^2 + 1^2 + 1^2 + 1^2\right) = \frac{9}{16} \times 4 = \frac{9}{4} \approx 2.25$

(a. iii) $y(t) = g(t)\cos(50t) = \frac{3}{2}\left(e^{j30^\circ}e^{j10t} + e^{-j30^\circ}e^{-j10t}\right)\frac{1}{2}\left(e^{j50t} + e^{-j50t}\right)$

$= \frac{3}{4}\left(e^{j30^\circ}e^{j60t} + e^{-j30^\circ}e^{j40t} + e^{j30^\circ}e^{-j40t} + e^{-j30^\circ}e^{-j60t}\right)$

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