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EES 351: Principles of Communications	**2020/1**

HW 6 — Due: Not Due **Solution**

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Problem 1. Consider the DSB-SC modem with no channel impairment shown in Figure 6.1. Suppose that the message is band-limited to $B = 3$ kHz and that $f_c = 100$ kHz.

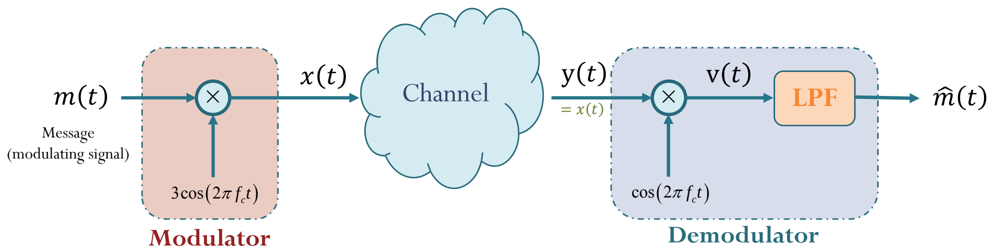
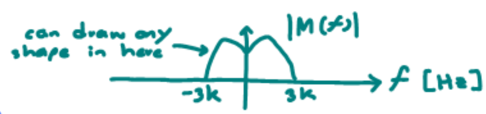


Figure 6.1: DSB-SC modem with no channel impairment

(a) Specify the frequency response $H_{LP}(f)$ of the LPF so that $\hat{m}(t) = m(t)$.

$M(f)$ is assumed to be band-limited to $B = 3$ kHz. Therefore, $M(f) = 0$ for $|f| > 3$ kHz:



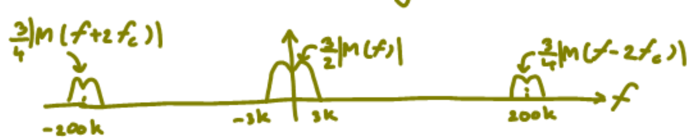
$$v(t) = y(t) \times \cos(2\pi f_c t) \overset{y(t) = x(t)}{=} x(t) \times \cos(2\pi f_c t) \overset{x(t) = m(t) \times 3\cos(2\pi f_c t)}{=} 3\, m(t) \cos^2(2\pi f_c t) = 3\, m(t) \times \frac{1}{2}\left(1 + \cos(2\pi(2f_c)t)\right)$$

$$= \frac{3}{2} m(t) + \frac{3}{2} m(t) \cos(2\pi(2f_c)t)$$

$$V(f) = \frac{3}{2} M(f) + \frac{3}{4} M(f - 2f_c) + \frac{3}{4} M(f + 2f_c).$$

Here, $f_c = 100$ kHz

Given the picture of $|M(f)|$ above, we now draw the corresponding $|V(f)|$:



To eliminate the terms $\frac{3}{4} M(f - 2f_c)$ and $\frac{3}{4} M(f + 2f_c)$, we set $H_{LP}(f) = 0$ for $|f| > 200\text{k} - 3\text{k} = 197$ kHz.

To allow $\frac{3}{2} M(f)$ to pass through, we set $H_{LP}(f) = c$ for $|f| < 3$ kHz for some constant c.

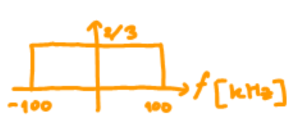
With such $H_{LP}(f)$, we get $\hat{m}(t) = c \times \frac{3}{2} m(t)$.

Because we need $\hat{m}(t) = m(t)$,

we have to set $\frac{3}{2} c = 1 \Rightarrow c = \frac{2}{3}$.

$$H_{LP}(f) = \begin{cases} \frac{2}{3}, & |f| \leq 3\text{ kHz}, \\ 0, & |f| \geq 197\text{ kHz}, \\ \text{any}, & \text{otherwise.} \end{cases}$$

An example would be $H_{LP}(f) = \begin{cases} 2/3, & |f| \leq 100\text{ kHz}, \\ 0, & \text{otherwise} \end{cases}$



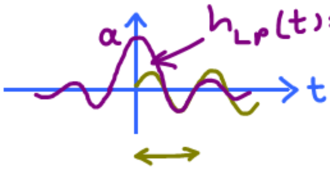
(b) Suppose the impluse response $h_{LP}(t)$ of the LPF is of the form $\alpha\,\mathrm{sinc}(\beta t)$. Find the constants α and β such that $\hat{m}(t) = m(t)$.

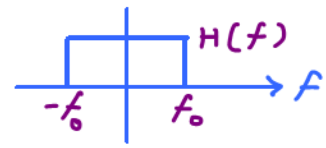
Observation:
The frequency of the sine wave in the sinc function fives the boundaries of the rectangular function in another domain.

$h_{LP}(t) = \alpha\,\mathrm{sinc}(\beta t)$, $\quad \beta t = 2\pi\left(\frac{\beta}{2\pi}\right)t$

$\frac{\beta}{2\pi}$: freq. of the sine wave

So, period $= \frac{2\pi}{\beta}$





period $= \frac{2\pi}{\beta}$

$\frac{1}{\text{width}} = \frac{\text{period}}{2} = \frac{\pi}{\beta}$

(of the rectangular pulse)

width $= \frac{\beta}{\pi}$

width $= 2f_0$

$f_0 = \frac{\beta}{2\pi}$

Area $= h_{LP}(0) = \alpha$

height $\times\, 2f_0 = \alpha$

height $= \frac{\alpha}{2f_0} = \frac{\alpha}{2\times\frac{\beta}{2\pi}} = \pi\frac{\alpha}{\beta}$

the gain of the LPF

From the previous part, we need $3 \leq f_0 < 197$ kHz.

$$\text{gain} = \frac{2}{3} \Rightarrow \pi\frac{\alpha}{\beta} = \frac{2}{3} \Rightarrow \alpha = \frac{2\beta}{3\pi}.$$

So, first we choose f_0. Then, we have $\beta = 2\pi f_0$ and $\alpha = \frac{2\beta}{3\pi}$.

An example would be $f_0 = 100$ kHz $\Rightarrow \beta = 2\times10^5\pi$ rad/s and $\alpha = \frac{2}{3\cancel{\pi}}\times 2\times10^5\cancel{\pi} = \frac{4}{3}\times10^5$

Alternatively, one may use $f_0 = B = 3$ kHz. Then, $\beta = 2\pi f_0 = 2\pi\times 3k = 6\times10^3\pi$ rad/s

$$\alpha = \frac{2}{3\pi}\beta = \frac{2}{\cancel{3}\cancel{\pi}}\times\cancel{6}^{2}\times10^3\cancel{\pi} = 4000.$$

Problem 2. This question starts with a ***square-modulator*** for DSB-SC . Then, the use of the square-operation block is further explored on the receiver side of the system. [Doerschuk, 2008, Cornell ECE 320]

(a) Let $x(t) = A_c m(t)$ where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$. Consider the block diagram shown in Figure 6.2.

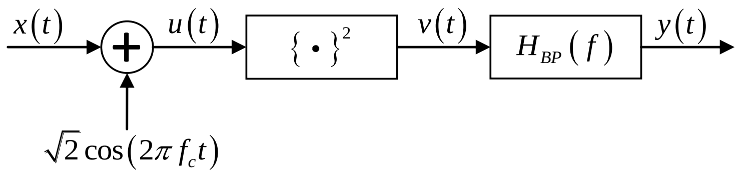


Figure 6.2: Block diagram for Problem 2a

Assume $f_c \gg B$ and

$$H_{BP}(f) = \begin{cases} 1, & |f - f_c| \le B \\ 1, & |f + f_c| \le B \\ 0, & \text{otherwise.} \end{cases}$$

The block labeled "$\{\cdot\}^2$" has output $v(t)$ that is the square of its input $u(t)$:

$$v(t) = u^2(t).$$

Find $y(t)$.

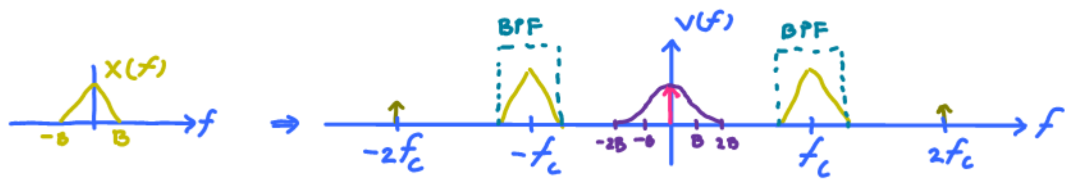
$x(t) = A_c m(t) \xrightarrow{\mathcal{F}} X(f) = A_c M(f)$. So, $X(f)$ is also bandlimited to B.

$u(t) = x(t) + \sqrt{2}\cos(\omega_c t)$ $\quad \omega_c = 2\pi f_c$

$v(t) = u^2(t) = (x(t) + \sqrt{2}\cos(\omega_c t))^2 = x^2(t) + 2\sqrt{2}x(t)\cos(\omega_c t) + \underbrace{2\cos^2(\omega_c t)}_{1+\cos(2\omega_c t)}$

$= 1 + x^2(t) + 2\sqrt{2}\, x(t)\cos(2\pi f_c t) + \cos(2\pi(2f_c)t)$

For example,



Note 1: $x^2(t) \xrightarrow{\mathcal{F}} X(f) * X(f)$. So, $x^2(t)$ is bandlimited to $2B$.
Because $f_c \gg B$, the spectrum of $x^2(t)$ will <u>not</u> be in the passband of the BPF which centers around f_c.

Note 2: The term $\cos(2\omega_c t)$ is at frequency $2\times f_c$ which again is <u>outside</u> the passband of the BPF.

Therefore, only the term $2\sqrt{2}\, x(t)\cos(2\pi f_c t)$ will survive the BPF.

$y(t) = \mathrm{BPF}\{v(t)\} = 2\sqrt{2}\, x(t)\cos\omega_c t = 2\sqrt{2} A_c m(t)\cos\omega_c t$

More generally, if the gain of the filter is g and the amplitude of the carrier is c,
then $y(t) = 2c \times g \times (\text{input signal}) \times \cos(\omega_c t)$
Here, $c = \sqrt{2}$, $g = 1$, and input signal $= A_c m(t)$.
Therefore, $y(t) = 2\sqrt{2} A_c m(t)\cos(\omega_c t)$.

(b) The block diagram in part (a) provides a nice implementation of a modulator because it may be easier to build a squarer than to build a multiplier. Based on the successful use of a squaring operation in the modulator, we decide to use the same squaring operation in the demodulator. Let

$$x(t) = A_c m(t) \sqrt{2} \cos(2\pi f_c t)$$

where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$. Again, assume $f_c \gg B$ Consider the block diagram shown in Figure 6.3.

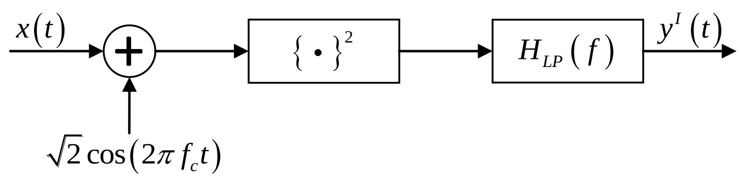


Figure 6.3: Block diagram for Problem 2b

Use

$$H_{LP}(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

Find $y^I(t)$. Does this block diagram work as a demodulator; that is, is $y^I(t)$ proportional to $m(t)$?

$$v(t) = \left(x(t) + \sqrt{2}\cos(\omega_c t)\right)^2 = 2\cos^2(\omega_c t)\left(A_c m(t) + 1\right)^2$$

$$= \underbrace{1 + \cos(2\omega_c t)\left(A_c^2 \underbrace{m^2(t)}_{\text{band-limited to } 2B} + 1 + 2A_c \underbrace{m(t)}_{\text{band-limited to } B}\right)}_{\text{Define this part as } g(t)} = g(t) + \underbrace{\overset{\text{LPF}}{\cancel{g(t)\cos(2\omega_c t)}}}_{0}$$

$g(t)\cos(2\omega_c t)$ is centered @ $2f_c$ and therefore will not pass though the LPF.

$$y^I(t) = \text{LPF}\{v(t)\} = \text{LPF}\{g(t)\} = 1 + 2A_c m(t) + \text{LPF}\{A_c^2 m^2(t)\}$$

This term has spectrum beyond IB. So, only a portion of it will pass through the LPF.

$y^I(t)$ is not proportional to $m(t)$.

Hence, this block diagram does not work as a demodulator.

(c) Due to the failure in part (b), we have to think hard and it seems natural to consider also the block diagram with cos replaced by sin. Let

$$x(t) = A_c m(t)\sqrt{2}\cos(2\pi f_c t)$$

where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$ as in part (b). Again, assume $f_c \gg B$ Consider the block diagram shown in Figure 6.4.

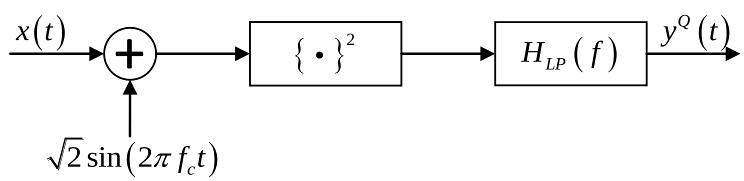


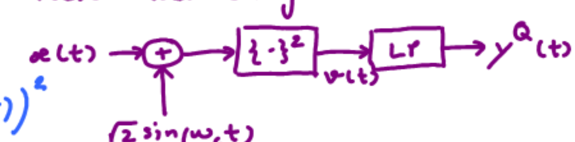
Figure 6.4: Block diagram for Problem 2c

As in part (b), use

$$H_{LP}(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

Find $y^Q(t)$.

New block diagram



Let $x(t) = A_c m(t)\sqrt{2}\cos(\omega_c t)$ as in part (b).

$v(t) = \left(x(t) + \sqrt{2}\sin(\omega_c t)\right)^2 = 2\left(A_c m(t)\cos(\omega_c t) + \sin(\omega_c t)\right)^2$

$= 2\left(A_c^2 m^2(t)\cos^2(\omega_c t) + A_c m(t)\underbrace{\cos(\omega_c t)\sin(\omega_c t)} + \sin^2(\omega_c t)\right)$

$2\cos\beta\sin\beta = \sin(2\beta)$

$= 2\left(A_c^2 m^2(t)\cos^2(\omega_c t) + \underbrace{\sin^2(\omega_c t)}_{= 1 - \cos^2 \omega_c t}\right) + A_c m(t)\sin(2\omega_c t)$

$= 2\left(\left(A_c^2 m^2(t) - 1\right)\cos^2(\omega_c t) + 1\right) + A_c m(t)\sin(2\omega_c t)$

$= 2 + \left(A_c^2 m^2(t) - 1\right)\left(1 + \cos(2\omega_c t)\right) + A_c m(t)\sin(2\omega_c t)$

LPF → 0   LPF → 0

$y^Q(t) = 2 + \mathrm{LPF}\{A_c^2 m^2(t)\} - 1 = \mathrm{LPF}\{A_c^2 m^2(t)\} + 1$

The output alone is far from being proportional to $m(t)$.
So, this block diagram also does not work as a demodulator.

(d) Use the results from parts (b) and (c). Draw a block diagram of a ***successful*** DSB-SC demodulator using squaring operations instead of multipliers.

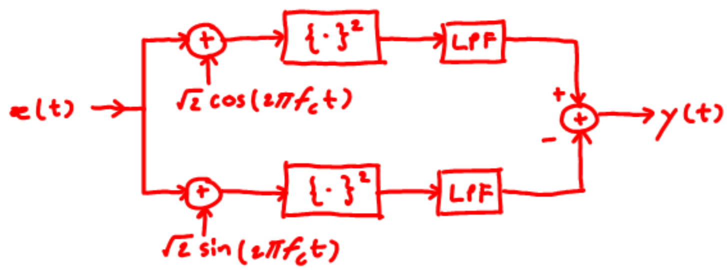
Observe that

$y^{I}(t) - y^{Q}(t) = 2A_c m(t)$

from (b) from (c)

which is the desired output of a successful DSB-SC demodulator.

Hence, the following block diagram would work:



Problem 3 (Cube modulator)**.** Consider the block diagram shown in Figure 6.5 where "$\{\cdot\}^3$" indicates a device whose output is the cube of its input.

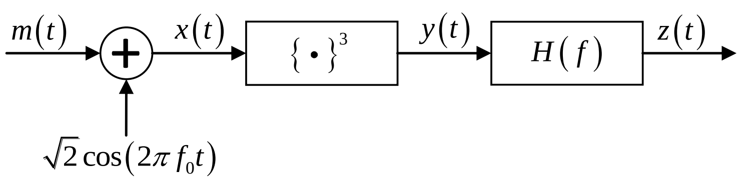


Figure 6.5: Block diagram for Problem 3. Note the use of f_0 instead of f_c.

Let $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ be bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$.

(a) Plot an $H(f)$ that gives $z(t) = m(t)\sqrt{2}\cos(2\pi f_c t)$. What is the gain in $H(f)$? What is the value of f_c? Notice that the frequency of the cosine is f_0 not f_c. You are supposed to determine f_c in terms of f_0.

$$y(t) = \left(m(t) + \sqrt{2}\cos(2\pi f_0 t)\right)^3 = m^3(t) + 3m^2(t)\sqrt{2}\cos\omega_0 t + \underbrace{3m(t)\,2\cos^2\omega_0 t}_{} + \underbrace{(\sqrt{2})^3\cos^3(\omega_0 t)}_{}$$

$$2\cos^2(\theta) = 1 + \cos(2\theta)$$

$$2\cos^3(\theta) = \cos\theta + \cos\theta\cos 2\theta = \cos\theta + \tfrac{1}{2}\cos\theta + \tfrac{1}{2}\cos 3\theta = \tfrac{3}{2}\cos\theta + \tfrac{1}{2}\cos 3\theta$$

$$= 3m(t)\left(1 + \cos 2\omega_0 t\right) = 3m(t) + 3m(t)\cos(2\omega_0 t)$$

$$= \frac{3}{\sqrt{2}}\cos(\omega_0 t) + \frac{1}{\sqrt{2}}\cos(3\omega_0 t)$$

We want $z(t) = m(t)\sqrt{2}\cos(\omega_c t)$. We see that the only term in $y(t)$ that has the form "constant × m × cos( )" is $3m(t)\cos(2\omega_0 t)$.

Therefore, we will center the passband to cover this part and adjust the gain to make the output the same as $z(t)$.

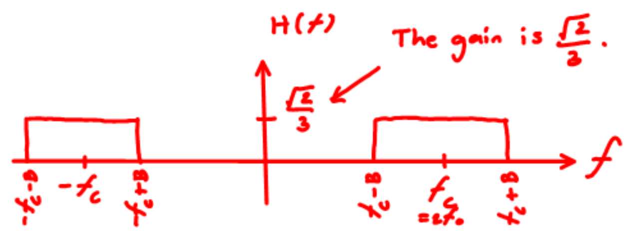
In particular, we need to make $2f_0 = f_c$. So, $f_0 = f_c/2$.

Let $H_{BP}(f) = \begin{cases} g, & |f - f_c| \le B, \\ g, & |f + f_c| \le B, \\ 0, & \text{otherwise.} \end{cases}$

Then, $z(t) = \underbrace{g \times 3}\, m(t)\cos(2\omega_0 t)$

we need $g \times 3 = \sqrt{2} \Rightarrow g = \frac{\sqrt{2}}{3}$

$\Rightarrow$

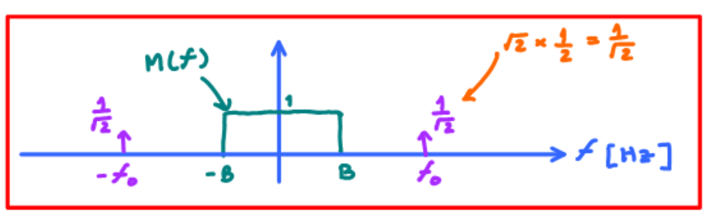


(b) Let $M(f)$ be

$$M(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

(i) Plot $X(f)$.

$x(t) = m(t) + \sqrt{2}\cos(2\pi f_0 t)$



(ii) Plot $Y(f)$. Hint:

$$M(f) * M(f) = \begin{cases} 2B - |f|, & |f| \le 2B \\ 0, & \text{otherwise.} \end{cases}$$

Do not attempt to make an accurate plot or calculation for the Fourier transform of $m^3(t)$.

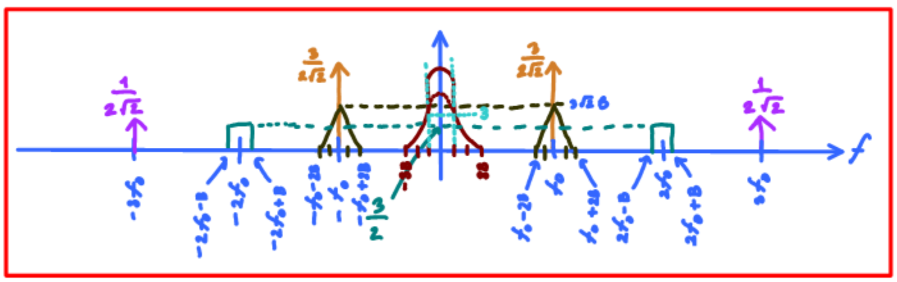
From (a), we have

$$y(t) = m^3(t) + 3m(t) + 3\sqrt{2}\, m^2(t)\cos(\omega_0 t) + 3m(t)\cos(2\omega_0 t) + \frac{1}{\sqrt{2}}\cos(3\omega_0 t) + \frac{3}{\sqrt{2}}\cos(\omega_0 t)$$

Without trying to make an accurate plot for $m^3(t)$, we know that it is band limited to $3B$.

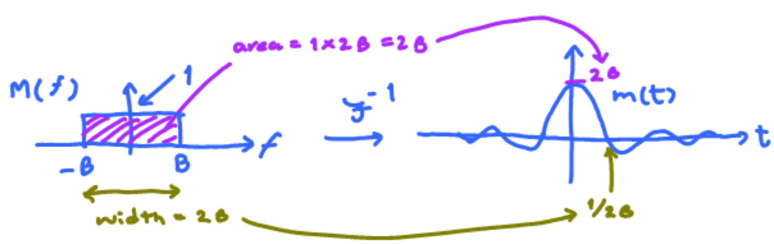
If you want to know the shape of $M(f)*M(f)*M(f)$, you can try plotting it in MATLAB using this code:

```
w = ones(1,10);
w2 = conv(w,w);
w3 = conv(w2,w);
plot(w3)
```



(iii) For your filter of part (a), plot $z(t)$.

$z(t) = m(t)\sqrt{2}\cos(2\pi f_c t)$. Therefore, to plot $z(t)$, first we need to find $m(t)$.

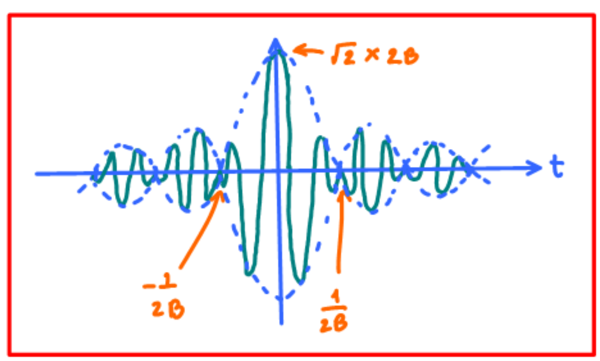


$z(t)$ is the above sinc. function multiplied by $\sqrt{2}\cos(2\pi f_c t)$.

Because $f_c \gg B$, we know that $\frac{1}{B} \gg \frac{1}{f_c}$

period of sine inside sinc period of cosine

So, the |sinc| function becomes the envelope of the cosine carrier.



[Doerschuk, 2008, Cornell ECE 320]