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EES 351: Principles of Communications **2020/1**

HW 3 — Due: September 11, 11:59 PM Solution

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 5 pages.

(b) (1 pt) Two choices for submission:

(i) Online submission via Google Classroom

- PDF only. Paper size should be the same as the posted file.
- Only for those who can directly work on the posted PDF file using devices with pen input.
- No scanned work, photos, or screen capture.
- Your file name should start with your 10-digit student ID: "5565242231 351 HW3.pdf"

(ii) Hardcopy submission: Work and write your answers **directly on a hardcopy of the posted file** (not on another blank sheet of paper).

(c) (1 pt) Write your first name and the last three digits of your student ID in the spaces provided on the upper-right corner of this page.

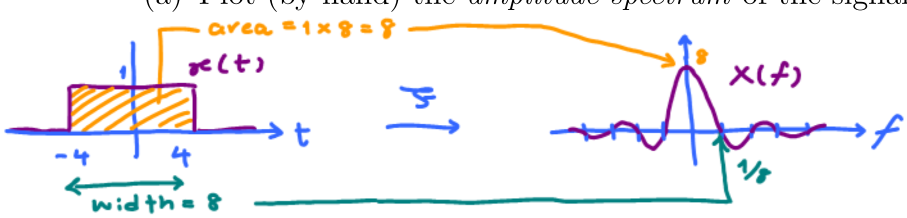
(d) (8 pt) Try to solve all problems.

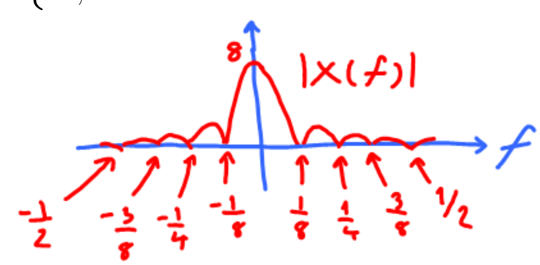
(e) Late submission will be heavily penalized.

Problem 1.

All of the signals under consideration here are rectangular pulses in the time domain. We know that the Fourier transform of an even rectangular pulse is a sinc function in the freq. domain.

(a) Plot (by hand) the *amplitude spectrum* of the signal $x(t) = \begin{cases} 1, & -4 < t < 4, \\ 0, & \text{otherwise.} \end{cases}$

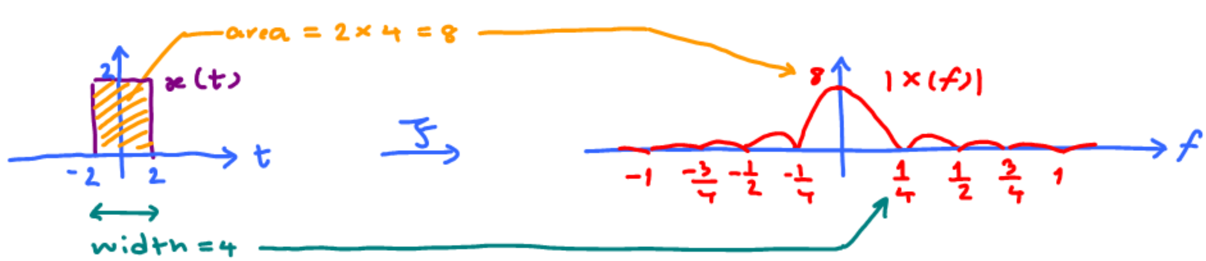




Note that we are not done yet. We are asked to plot the amplitude spectrum (|X(f)|) and not the Fourier transform X(f) itself. Because x(f) is real-valued, |X(f)| is simply the absolute value of X(f). So, the "negative" part of X(f), gets rectified (flipped up).

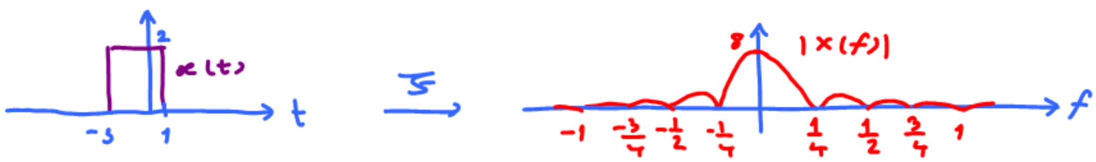
(b) Plot (by hand) the *amplitude spectrum* of the signal $x(t) = \begin{cases} 2, & -2 < t < 2, \\ 0, & \text{otherwise.} \end{cases}$

Here, we simply follow the same technique that was applied in part (a)



(c) Plot (by hand) the *amplitude spectrum* of the signal $x(t) = \begin{cases} 2, & -3 < t < 1, \\ 0, & \text{otherwise.} \end{cases}$

Note that the signal x(t) in this part is the same aa the one in part (b) but shifted to the left. We have seen in class that time-shifting of the whole signal does not change its amplitude spectrum. Therefore, we can simply copy the plot of |X(f)| from part (b) here.



Problem 2.

(a) Suppose the Fourier transform of a signal $x(t)$ is given by

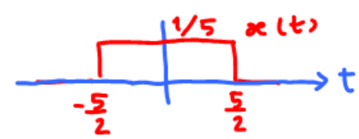
$$X(f) = \text{sinc}(5\pi f) = \frac{\sin(5\pi f)}{(5\pi f)} \stackrel{①}{=} \frac{\sin\left(2\pi \frac{5}{2} f\right)}{5\pi f}.$$

(i) Plot (by hand) $x(t)$. ⑤

"period" = $\frac{2}{5}$ ③ ⇐ "freq" = $\frac{5}{2}$ ②

④ $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5} = \frac{1}{\text{width of } \sqcap}$

So, the rectangular pulse in the tine domain has width = 5.
Its area under the graph must be 1. So, its height must be 1/5.



(ii) Find $\int_{-\infty}^{\infty} X(f)df$. (Hint: This integration is exactly the inverse Fourier transform formula with $t = 0$.)

$$\int_{-\infty}^{\infty} X(f)\,df = x(0) = \frac{1}{5}.$$

(b) Suppose the Fourier transform of a signal $y(t)$ is given by

$$Y(f) = \operatorname{sinc}^2(5\pi f) = \left(\frac{\sin(5\pi f)}{(5\pi f)}\right)^2.$$

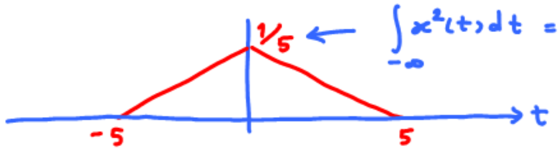
(i) Plot (by hand) $y(t)$.

Note that $Y(f) = (X(f))^2 = X(f) \underbrace{\times}_{\text{multiplication}} X(f)$.

By the convoltuion property of Fourier transform, we know that

$$y(t) = x(t) \underbrace{*}_{\text{convolution.}} x(t)$$

← We discussed convolution of rect. functions in class.



$\int_{-\infty}^{\infty} x^2(t)\,dt = \left(\frac{1}{5}\right)^2 \times 5 = \frac{1}{5}$

(ii) Find $\int_{-\infty}^{\infty} Y(f)df$. $= y(0) = \frac{1}{5}$.

[See the added extra page for alternative solutions.]

Problem 3. The Fourier transform of the triangular pulse $g(t)$ in Figure 3.1a is given as

$$G(f) = \frac{1}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right)$$

Using this information, and the time-shifting and time-scaling properties, find the Fourier transforms of the signals shown in Figure 3.1b, c, d, e, and f.

Remark: Don't forget to simplify your answers. For example, the answer in part (d) should be of the form $\operatorname{sinc}^2(\cdot)$ and the answer in part (e) should be of the form $\operatorname{sinc}(\cdot)$

(b) Note that $g_1(t) = g(-t)$.

Recall that $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$.

Here, $a = -1$.

Therefore, $G_1(f) = \frac{1}{|-1|} G\left(\frac{f}{-1}\right) = \frac{1}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right)$

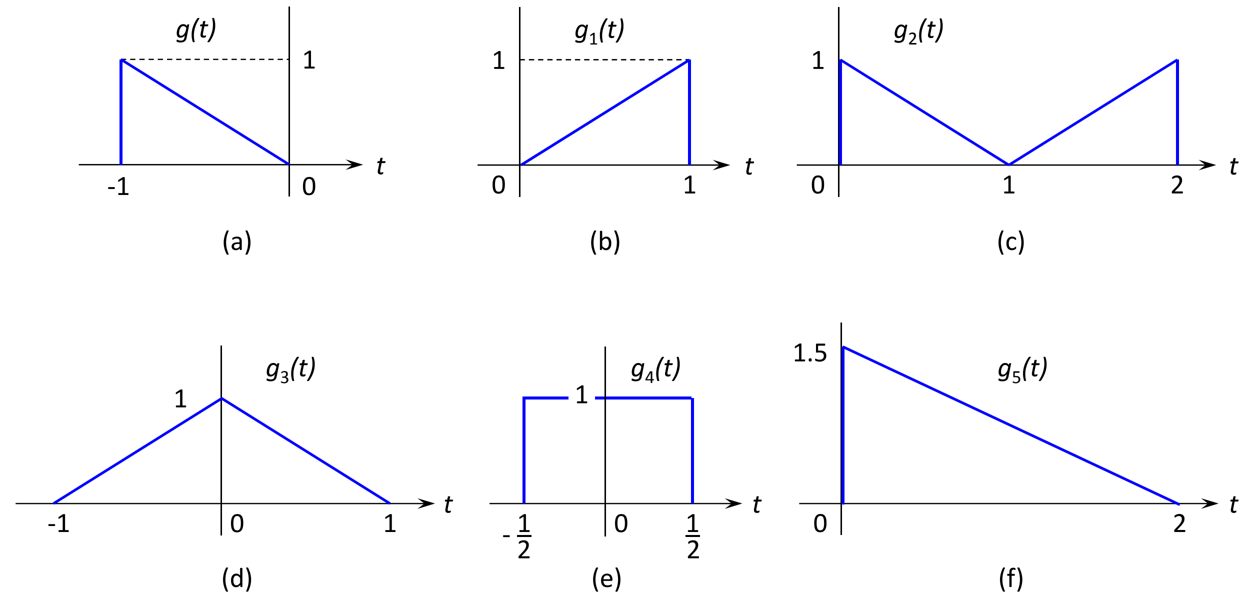


Figure 3.1: Problem 3

(c) Note that $g_2(t) = g(t-1) + g_1(t-1)$

$$\Rightarrow \quad G_2(f) = e^{-j2\pi f} G(f) + e^{-j2\pi f} G_1(f) = \frac{e^{-j\omega}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1 + e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

Here, we write "ω" instead of "$2\pi f$".
After we're done massaging the expressions, we change "ω" back to "$2\pi f$"

$$\frac{e^{-j\omega}}{\omega^2}\left(2\cos(\omega) - j\omega(2j)\sin\omega - 2\right)$$

$$= \frac{2e^{-j2\pi f}}{(2\pi f)^2}\left(\cos(2\pi f) + 2\pi f \sin(2\pi f) - 1\right)$$

(d) Note that $g_3(t) = g(t-1) + g_1(t+1)$

$$\Rightarrow \quad G_3(f) = e^{-j2\pi f} G(f) + e^{j2\pi f} G_1(f) = \frac{1}{\omega^2}\left(1 - \cancel{j\omega} - e^{-j\omega} + 1 + \cancel{j\omega} - e^{j\omega}\right)$$

$$= \frac{1}{\omega^2}\left(2 - e^{-j\omega} - e^{j\omega}\right) = -\frac{1}{\omega^2}\left(e^{j\omega} - 2 + e^{-j\omega}\right) = -\frac{1}{\omega^2}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)^2$$

$$= -\frac{1}{\omega^2}\left(2j\sin\frac{\omega}{2}\right)^2 = \frac{\sin^2\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)^2} = \text{sinc}^2\left(\frac{\omega}{2}\right) = \text{sinc}^2(\pi f)$$

(e) Note that $g_4(t) = g(t-\frac{1}{2}) + g_1(t+\frac{1}{2})$.

use part (b)

$$\Rightarrow \quad G_4(f) = e^{-j\omega/2} G(f) + e^{j\omega/2} G_1(f)$$

$$= e^{-j\omega/2} \frac{1}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right) + e^{j\omega/2} \frac{1}{\omega^2}\left(e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{1}{\omega^2}\left(\cancel{e^{j\frac{\omega}{2}}} - j\omega e^{j\frac{\omega}{2}} - \cancel{e^{-j\frac{\omega}{2}}} + \cancel{e^{-j\frac{\omega}{2}}} + j\omega e^{-j\frac{\omega}{2}} - \cancel{e^{j\frac{\omega}{2}}}\right) = \frac{-j}{\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

$$= \frac{(-j)}{\omega}(2j)\sin(\omega/2) = \frac{\sin(\omega/2)}{\omega/2} = \operatorname{sinc}\left(\frac{\omega}{2}\right) = \operatorname{sinc}(\pi f)$$

(f) Note that $g_5(t) = 1.5\, g\left(\frac{1}{2}(t-2)\right)$

$$\Rightarrow \quad G_5(f) = 1.5 \times \frac{1}{1/2} G\left(\frac{f}{1/2}\right) e^{-j2\omega} = 3G(2f)e^{-j}$$

$$= 3 \times \frac{1}{(2\pi 2f)^2}\left(e^{j2\omega} - j2\omega e^{j2\omega} - 1\right)e^{-j2\omega} = \frac{3}{4\omega^2}\left(1 - 2j\omega - e^{-2j\omega}\right)$$

$$= \frac{3}{4(2\pi f)^2}\left(1 - j4\pi f - e^{-j4\pi f}\right)$$

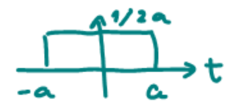
[Alternative solutions for Problem 2]

Alternatively, we can try to solve (a.i) and (b.i) via formula.

(a.i) We know that

$$2a\,\text{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} 1[|t| \le a]$$ ← shown in class.

Therefore, $$\text{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \le a].$$



Here, $2a = 5$. So, $a = 5/2$.

(b.i)

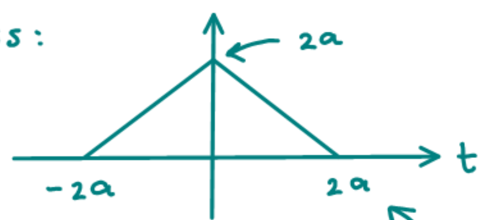
$$\text{sinc}^2(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \le a] * \frac{1}{2a} 1[|t| \le a]$$

$$= \frac{1}{4a^2}\left(1[|t| \le a] * 1[|t| \le a]\right)$$

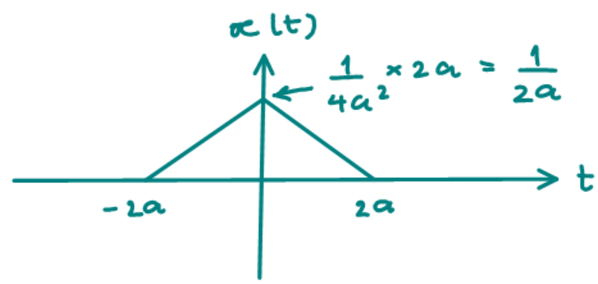
So, we can solve this question if we can find the convolution of $1[|t| \le a]$ with itself.

This is also discussed in class:

$$1[|t| \le a] * 1[|t| \le a] =$$



Therefore, the plot of $x(t)$ should be the same as but scaled vertically by a factor of $1/4a^2$:



Alternatively, one can use the Parseval's theorem to solve (b.ii):

$$\int_{-\infty}^{\infty} Y(f)\,df = \int_{-\infty}^{\infty} X^2(f)\,df = \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{5}.$$

Note that we don't have to write $|\cdot|$ because $x(t)$ and $X(f)$ are real-valued.