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EES 351: Principles of Communications **2020/1**

HW 2 — Due: September 2, 11:59 PM **Solution**

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Instructions

(a) This assignment has 5 pages.

(b) (1 pt) Two choices for submission:

(i) Online submission via Google Classroom

- PDF only. Paper size should be the same as the posted file.
- Only for those who can directly work on the posted PDF file using devices with pen input.
- No scanned work, photos, or screen capture.
- Your file name should start with your 10-digit student ID: "5565242231 315 HW1.pdf"

(ii) Hardcopy submission: Work and write your answers **directly on a hardcopy of the posted file** (not on another blank sheet of paper).

(c) (1 pt) Write your first name and the last three digits of your student ID in the spaces provided on the upper-right corner of this page.

(d) (8 pt) Try to solve all problems.

(e) [ENRpr] = Explanation is not required for this problem.

(f) Late submission will be heavily penalized.

Problem 1. In class, we have seen how to use the Euler's formula to show that

$$\cos^2 x = \frac{1}{2}\left(\cos\left(2x\right) + 1\right).$$

For this question, ***apply similar technique*** to show that

$$\cos A \cos B = \frac{1}{2}\left(\cos\left(A+B\right) + \cos\left(A-B\right)\right).$$

$$\cos A \cos B = \frac{1}{2}\left(e^{jA}+e^{-jA}\right) \times \frac{1}{2}\left(e^{jB}+e^{-jB}\right)$$

$$= \left(e^{jA}+e^{-jA}\right)\left(e^{jB}+e^{-jB}\right) \times \frac{1}{4}$$

$$= \Big(\underbrace{e^{j(A+B)}+e^{-j(A+B)}}_{2\cos(A+B)}+\underbrace{e^{j(A-B)}+e^{-j(A-B)}}_{2\cos(A-B)}\Big)\frac{1}{4}$$

$$= \frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$$

Steps:

① Replace cos and sin with complex exponential functions.

② Simplify or rearrange the expression

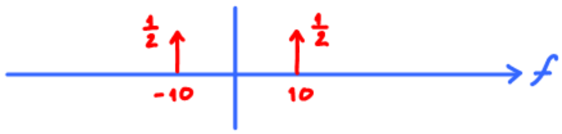
③ Convert back to cos and sin

First, we convert the given expressions into complex exponential functions. Then, we use the fact that $e^{j2\pi f_0 t}$ in the time domain corresponds to the delta function at $f = f_0$ in the frequency domain

Problem 2. Plot (by hand) the Fourier transforms of the following signals

(a) $\cos(\underbrace{20\pi t}_{A = 2\pi(10)t}) = \frac{e^{jA} + e^{-jA}}{2} = \frac{1}{2}e^{jA} + \frac{1}{2}e^{-jA} = \frac{1}{2}e^{j2\pi(10)t} + \frac{1}{2}e^{j2\pi(-10)t}$.

So, the plot of its Fourier transform is



Alternatively, one may simply remember that the Fourier transform of $\cos(2\pi f_0 t)$ is simply delta functions of size $\frac{1}{2}$ at f_0 and $-f_0$.

(b) $\cos(20\pi t) + \cos(40\pi t)$

For $\cos(40\pi t)$, the corresponding frequencies are ± 20 Hz.

$2\pi f_0 t = 40\pi t$

$f_0 = 20$

So, the plot of the Fourier transform of $\cos(20\pi t) + \cos(40\pi t)$ is

already got from part (a)

(c) $(\cos(20\pi t))^2$

$(\cos(\underbrace{20\pi t}_{A = 20\pi t = 2\pi(10)t}))^2 = (\cos A)^2 = \left(\frac{1}{2}(e^{jA} + e^{-jA})\right)^2 = \frac{1}{4}\left(e^{2jA} + 2 + e^{-2jA}\right)$

$= \frac{1}{4}e^{j2\pi(20)t} + \frac{1}{2}\underbrace{e^{j2\pi(0)t}}_{1} + \frac{1}{4}e^{j2\pi(-20)t}$

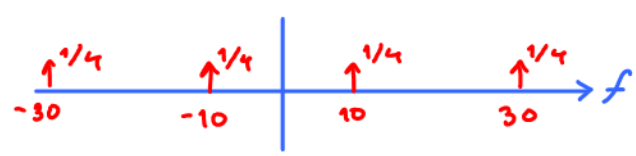
So, the plot of its Fourier transform is

(d) $\cos(\underbrace{20\pi t}_{A = 20\pi t = 2\pi(10)t}) \times \cos(\underbrace{40\pi t}_{B = 40\pi t = 2\pi(20)t}) = \cos(A)\cos(B) = \frac{1}{2}(e^{jA} + e^{-jA})\frac{1}{2}(e^{jB} + e^{-jB})$

$= \frac{1}{4}\left(e^{j(A+B)} + e^{j(-A+B)} + e^{j(A-B)} + e^{j(-A-B)}\right)$

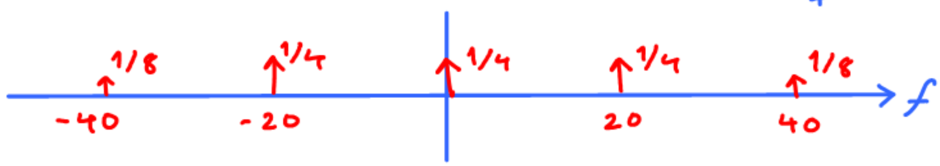
$= \frac{1}{4}\left(e^{j2\pi(30)t} + e^{j2\pi(10)t} + e^{j2\pi(-10)t} + e^{j2\pi(-30)t}\right)$

So, the plot of its Fourier transform is



(e) $(\cos(20\pi t))^2 \times \cos(40\pi t) = \Big(\underbrace{\tfrac{1}{4}e^{j2\pi(20)t} + \tfrac{1}{2} + \tfrac{1}{4}e^{j2\pi(-20)t}}_{\text{from part (c)}}\Big) \times \Big(\tfrac{1}{2}e^{j2\pi(20)t} + \tfrac{1}{2}e^{j2\pi(-20)t}\Big)$

$= \frac{1}{8}e^{j2\pi(40)t} + \frac{1}{4}e^{j2\pi(20)t} + \underbrace{\frac{1}{8}e^{0} + \frac{1}{8}e^{0}}_{\frac{1}{4}e^{j2\pi(0)t}} + \frac{1}{4}e^{j2\pi(-20)t} + \frac{1}{8}e^{j2\pi(-40)t}$



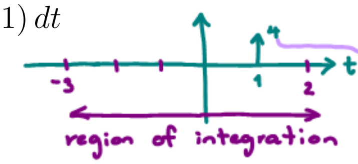
Problem 3. Evaluate the following integrals:

(a) First, recall that $\int_A \delta(t)\,dt = \begin{cases} 1, & 0 \in A, \\ 0, & 0 \notin A. \end{cases}$ In particular, $\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$

(i) $\int\limits_{-\infty}^{\infty} 2\delta(t)\,dt = 2\int_{-\infty}^{\infty} \delta(t)\,dt = 2 \times 1 = 2.$

(ii) $\int\limits_{-3}^{2} 4\delta(t-1)\,dt$

Consider the function $4\delta(t-1)$ graphically.

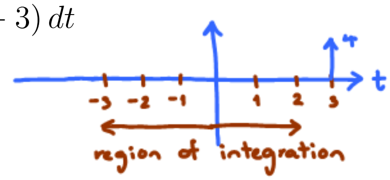


The area under the curve from -3 to 2 includes the arrow area which is 4.

So, $\int_{-3}^{2} 4\delta(t-1)\,dt = 4.$

(iii) $\int\limits_{-3}^{2} 4\delta(t-3)\,dt$

Consider the function $4\delta(t-3)$ graphically.



The area under the curve from -3 to 2 does not include the arrow area.

Therefore, $\int_{-3}^{2} 4\delta(t-3)\,dt = 0.$

(b) $\int\limits_{-\infty}^{\infty} \delta(t)\,\underbrace{e^{-j2\pi ft}}\,dt = \int_{-\infty}^{\infty} g(t)\,\delta(t)\,dt \overset{\text{sifting property}}{=} g(0) = e^{-j2\pi ft}\Big|_{t=0} = e^{0} = 1$

(c)

(i) $\int\limits_{-\infty}^{\infty} \delta(t-2)\,\underbrace{\sin(\pi t)}\,dt = \int_{-\infty}^{\infty} g(t)\,\delta(t-2)\,dt \overset{\text{sifting property v2}}{=} g(2) = \sin(\pi t)\Big|_{t=2} = \sin(2\pi) = 0$

sifting property v2

(ii) $\int\limits_{-\infty}^{\infty} \delta(t+3)\, e^{-t} dt = \int\limits_{-\infty}^{\infty} g(t)\, \delta(t-(-3))\, dt = g(-3) = e^{-t}\Big|_{t=-3} = e^{-(-3)} = e^{3}$

sifting property v2

(iii) $\int\limits_{-\infty}^{\infty} e^{(x-1)} \cos\left(\frac{\pi}{2}(x-5)\right) \delta(x-3) dx = \int\limits_{-\infty}^{\infty} g(t)\, \delta(x-3)\, dx = g(3) = e^{x-1} \cos\left(\frac{\pi}{2}(x-5)\right)\Big|_{x=3} = e^{2} \cos(-\pi) = -e^{2}$

Note that the "x" here is just a dummy variable. It takes the role of "t" in our formula.

(d) This part has the delta function in the form $\delta(T-t)$.

We use the "change of variables" technique to evaluate the integral: $\int\limits_{-\infty}^{\infty} g(t)\, \delta(T-t)\, dt = -\int\limits_{\infty}^{-\infty} g(T-\tau)\, \delta(\tau)\, d\tau = \int\limits_{-\infty}^{\infty} g(T-t)\, \delta(t)\, dt = g(T-0) = g(T).$

change of variables $\begin{cases} \tau = T-t \\ t = T-\tau \\ dt = -d\tau \end{cases}$ $\begin{cases} t = \tau \\ dt = d\tau \end{cases}$

(i) $\int\limits_{-\infty}^{\infty} (t^3+4)\, \delta(1-t) dt = t^3+4\Big|_{t=1} = 1^3+4 = 1+4 = 5$

(ii) $\int\limits_{-\infty}^{\infty} g(2-t)\, \delta(3-t) dt = g(2-t)\Big|_{t=3} = g(2-3) = g(-1)$

Remark: From $\int\limits_{-\infty}^{\infty} g(t)\, \delta(T-t)\, dt = g(T)$, we get $\underbrace{\int\limits_{-\infty}^{\infty} g(\tau)\, \delta(t-\tau)\, d\tau}_{(g*\delta)(t)} = g(t)$ simply by variable renaming $\left(\begin{smallmatrix} t \to \tau \\ T \to t \end{smallmatrix}\right)$

(e) $\int\limits_{-2}^{2} \delta(2t)\, dt$

$= \int\limits_{-4}^{4} \delta(x) \frac{1}{2} dx = \frac{1}{2} \times \int\limits_{-4}^{4} \delta(x)\, dx = \frac{1}{2} \times 1 = \frac{1}{2}.$

change of variables $\begin{cases} x = 2t \\ t = \frac{1}{2}x \\ dt = \frac{1}{2}dx \end{cases}$

$0 \in [-4,4]$

Alternatively, we know that $\delta(at) = \frac{1}{|a|}\delta(t)$.

Therefore, $\delta(2t) = \frac{1}{2}\delta(t)$.

Hence, $\int\limits_{-2}^{2} \delta(2t)\, dt = \int\limits_{-2}^{2} \frac{1}{2}\delta(t)\, dt = \frac{1}{2}\int\limits_{-2}^{2} \delta(t)\, dt = \frac{1}{2} \times 1 = \frac{1}{2}.$

$0 \in [-2,2]$

Problem 4. Consider the signal $g(t)$ shown in Figure 2.1.

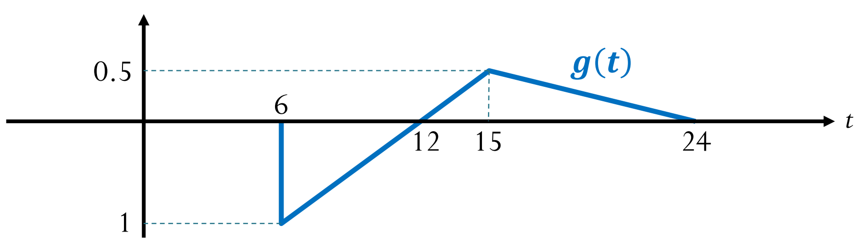


Figure 2.1: Problem 4

(a) Carefully sketch the following signals:

(i) $y_1(t) = g(-t)$

(ii) $y_2(t) = g(t+6)$

(a)

(i) Recall the time inversion (time reversal) operation

$g(-t)$ is the mirror image of $g(t)$ about the vertical axis.

(ii) Recall the time shifting operation:

$g(t-T)$ represents $g(t)$ time-shifted by T.

If T is positive, the shift is to the right (delay).

If T is negative, the shift is to the left (by $|T|$).

Here, $y_2(t) = g(t+6) = g(t-(-6))$.

So, $y_2(t)$ is simply $g(t)$ shifted to the left by 6 time units.

(iii) Recall the time scaling operation:

$g(at)$ is $g(t)$ compressed in time by the factor a.

for $a>1$

So, $y_3(t) = g(3t)$ is simply $g(t)$ compressed in time by a factor of 3.

(iv) The tricky one would be $g(6-t)$.

There are two ways to think about it

① $g(t) \xrightarrow{\text{time inversion}} g(-t) \xrightarrow{\text{time shift, } T=6} g(-(t-6))$

$g(-t)$: mirror image about the vertical axis

$g(-(t-6))$: shift to the right by 6

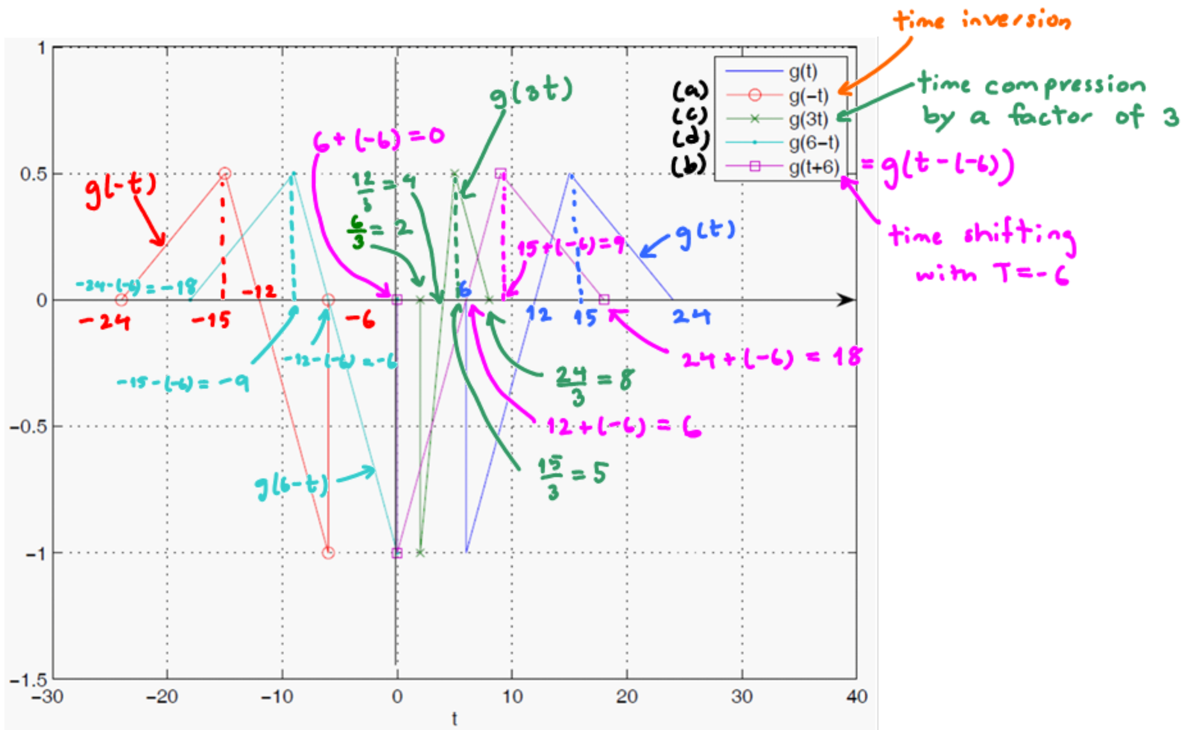
② $g(t) \xrightarrow{\text{time shift, } T=-6} g(t+6) \xrightarrow{\text{time inversion}} g(-t+6)$

$g(t+6)$: shift to the left by 6

$g(-t+6)$: mirror image of $g(t+6)$ about the vertical axis

(iii) $y_3(t) = g(3t)$

(iv) $y_4(t) = g(6 - t)$.



(b) Find the "net" area under the graph for each of the signals in the previous part. (Mathematically, this is equivalent to integrating each signal from $-\infty$ to $+\infty$. However, directly calculating and combining positive and negative areas from the plots should be easier.) First, note that, for any constant m, c,

$$\int_{-\infty}^{\infty} g(mt+c)\,dt = \begin{cases} \xrightarrow{m>0} \int_{-\infty}^{\infty} g(x)\frac{1}{m}\,dx = \frac{1}{m}\int_{-\infty}^{\infty} g(x)\,dx \\ \xrightarrow{m<0} \int_{\infty}^{-\infty} g(x)\frac{1}{m}\,dx = -\frac{1}{m}\int_{-\infty}^{\infty} g(x)\,dx \end{cases} = \frac{1}{|m|}\int_{-\infty}^{\infty} g(x)\,dx$$

$x = mt + c$

$dx = m\,dt$

$dt = \frac{1}{m}\,dx$

Now, for us, $\int_{-\infty}^{\infty} g(t)\,dt = \underbrace{-\left(\frac{1}{2}\times 1 \times \overset{12-6}{6}\right)}_{\text{area under the first triangle}} + \left(\frac{1}{2}\times\frac{1}{2}\times \overset{24-12}{12}\right) = -3+3 = 0.$

Therefore, $\int_{-\infty}^{\infty} g(mt+c)\,dt = 0$ for any m, c.

Note:

	m	c
(i)	-1	0
(ii)	1	6
(iii)	0	0
(iv)	-1	6