Name ________________________ ID3 _ _ _

EES 315: Probability and Random Processes **2020/1**

HW 8 — Due: November 13, 11:59 PM

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 6 pages.

(b) (1 pt) Two choices for submission:

(i) Online submission via Google Classroom

- PDF only. Paper size should be the same as the posted file.
- Only for those who can directly work on the posted PDF file using devices with pen input.
- No scanned work, photos, or screen capture.
- Your file name should start with your 10-digit student ID: "5565242231 315 HW8.pdf"

(ii) Hardcopy submission: Work and write your answers **directly on a hardcopy of the posted file** (not on another blank sheet of paper).

(c) (1 pt) Write your first name and the last three digits of your student ID in the spaces provided on the upper-right corner of this page.

(d) (8 pt) Try to solve all problems. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

(e) Late submission will be heavily penalized.

Problem 1. The random variable V has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P\left[V \in \{u^2 : u = 1, 2, 3, \ldots\}\right]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

(e) Sketch $p_V(v)$.

(f) Sketch $F_V(v)$. (Note that $F_V(v) = P[V \leq v]$.)

Problem 2. The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8}, \\ 0.2, & \frac{1}{8} \leq x < \frac{1}{4}, \\ 0.9, & \frac{1}{4} \leq x < \frac{3}{8}, \\ 1 & x \geq \frac{3}{8}. \end{cases}$$

Determine the following probabilities:

(a) $P[X \leq 1/18]$

(b) $P[X \leq 1/4]$

(c) $P[X \leq 5/16]$

(d) $P[X > 1/4]$

(e) $P[X \leq 1/2]$

[Montgomery and Runger, 2010, Q3-42]

Remark: Try to calculate these values directly from the cdf. (Avoid converting the cdf to pmf first.)

Problem 3. [F2013/1] For each of the following random variables, find $P[1 < X \leq 2]$.

(a) $X \sim \text{Binomial}(3, 1/3)$

(b) $X \sim \text{Poisson}(3)$

Problem 4. Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 2$ customers per minute. Let M be the number of customers arriving between 9:00 and 9:05. What is the probability that $M < 2$?

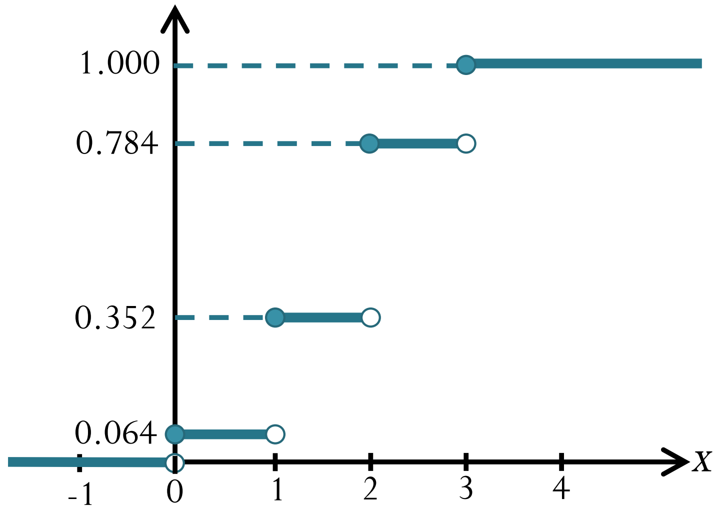

Figure 8.1: CDF of X for Problem 5

Problem 5. [M2011/1] The cdf of a random variable X is plotted in Figure 8.1.

(a) Find the pmf $p_X(x)$.

(b) Find the family to which X belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

Problem 6. When n is large, binomial distribution Binomial(n, p) becomes difficult to compute directly . In this question, we will consider an approximation when the value of p

is close to 0. In such case, the binomial can be approximated[1] by the Poisson distribution with parameter $\alpha = np$. For this approximation to work, we will see in this exercise that n does not have to be very large and p does not need to be very small.

(a) Let $X \sim \text{Binomial}(12, 1/36)$. (For example, roll two dice 12 times and let X be the number of times a double 6 appears.) Evaluate $p_X(x)$ for $x = 0, 1, 2$.

(b) Compare your answers part (a) with its Poisson approximation.

Problem 7. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

[1] More specifically, suppose X_n has a binomial distribution with parameters n and p_n. If $p_n \to 0$ and $np_n \to \alpha$ as $n \to \infty$, then

$$P[X_n = k] \to e^{-\alpha}\frac{\alpha^k}{k!}.$$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 8. A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

(a) exactly one is emitted,

(b) more than three are emitted,

(c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

Problem 9 (M2011/1)**.** You are given an unfair coin with probability of obtaining a heads equal to $1/3,000,000,000$. You toss this coin 6,000,000,000 times. Let A be the event that you get "tails for all the tosses". Let B be the event that you get "heads for all the tosses".

(a) Approximate $P(A)$.

(b) Approximate $P(A \cup B)$.