EES 315: Probability and Random Processes **2020/1**

HW Solution 7 — Due: November 6, 11:59 PM

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Problem 1 (Majority Voting in Digital Communication)**.** A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a "codeword" 111 is transmitted, and to send the message 0, a "codeword" 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Let $\mathcal{E}$ be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event $\mathcal{E}$ occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2}p^2(1-p) + \binom{3}{3}p^3 = p^2(3-2p).$$

When $p = 0.1$, we have $P(\mathcal{E}) \approx \boxed{0.028}$.

Problem 2. For each description of a random variable X below, indicate whether X is a **discrete** random variable.

(a) X is the number of websites visited by a randomly chosen software engineer in a day.

(b) X is the number of classes a randomly chosen student is taking.

(c) X is the average height of the passengers on a randomly chosen bus.

(d) A game involves a circular spinner with eight sections labeled with numbers. X is the amount of time the spinner spins before coming to a rest.

(e) X is the thickness of the longest book in a randomly chosen library.

(f) X is the number of keys on a randomly chosen keyboard.

(g) X is the length of a randomly chosen person's arm.

Solution:We consider the number of possibilities for the values of X in each part. If the collection of possible values is countable (finite or countably infinite), then we conclude that the random variable is discrete. Otherwise, the random variable is not discrete. Therefore, the X defined in parts (a), (b), and (f) are discrete. The X defined in other parts are not discrete.

Problem 3 (Quiz4, 2014)**.** Consider a random experiment in which you roll a 20-sided fair dice. We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega, \quad Y(\omega) = (\omega - 5)^2, \quad Z(\omega) = |\omega - 5| - 3$$

Evaluate the following probabilities:

(a) $P[X = 5]$

(b) $P[Y = 16]$

(c) $P[Y > 10]$

(d) $P[Z > 10]$

(e) $P[5 < Z < 10]$

Solution: In this question, $\Omega = \{1, 2, 3, \ldots, 20\}$ because the dice has 20 sides. All twenty outcomes are equally-likely because the dice is fair. So, the probability of each outcome is $\frac{1}{20}$:

$$P(\{\omega\}) = \frac{1}{20} \text{ for any } \omega \in \Omega.$$

(a) From $X(\omega) = \omega$, we have $X(\omega) = 5$ if and only if $\omega = 5$.

Therefore, $P[X = 5] = P(\{5\}) = \boxed{\frac{1}{20}}$.

(b) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) = 16$ if and only if $\omega = \pm 4 + 5 = 1$ or 9.

Therefore, $P[Y = 16] = P(\{1, 9\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(c) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) > 10$ if and only if $(\omega - 5)^2 > 10$.
To check this, it may be more straight-forward to calculate $Y(\omega)$ at all possible values of ω:

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Y(\omega)$	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

From the table, the values of ω that satisfy the condition $Y(\omega) > 10$ are $1, 9, 10, 11, \ldots, 20$.

Therefore, $P[Y > 10] = P(\{1, 9, 10, 11, \ldots, 20\}) = \boxed{\frac{13}{20}}$.

(d) The values of ω that satisfy $|\omega - 5| - 3 > 10$ are 19 and 20.
To see this, it is straight-forward to calculate $Z(\omega)$ at all possible values of ω:

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Z(\omega)$	1	0	-1	-2	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12

Therefore, $P[Z > 10] = P(\{19, 20\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(e) The values of ω that satisfy $5 < |\omega - 5| - 3 < 10$ are $14, 15, 16, 17$.
Therefore, $P[5 < Z < 10] = P(\{14, 15, 16, 17\}) = \frac{4}{20} = \boxed{\frac{1}{5}}$.

Problem 4. Consider the sample space $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$. Suppose that $P(A) = |A|/|\Omega|$ for any event $A \subset \Omega$. Define the random variable $X(\omega) = \omega^2$. Find the probability mass function of X.

Solution: The random variable maps the outcomes $\omega = -2, -1, 0, 1, 2, 3, 4$ to numbers $x = 4, 1, 0, 1, 4, 9, 16$, respectively. Therefore,

$$\begin{aligned} p_X(0) &= P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7}, \\ p_X(1) &= P(\{\omega : X(\omega) = 1\}) = P(\{-1, 1\}) = \frac{2}{7}, \\ p_X(4) &= P(\{\omega : X(\omega) = 4\}) = P(\{-2, 2\}) = \frac{2}{7}, \\ p_X(9) &= P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and} \\ p_X(16) &= P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}. \end{aligned}$$

Combining the results above, we get the complete pmf:

$$p_X(x) = \boxed{\begin{cases} \frac{1}{7}, & x = 0, 9, 16, \\ \frac{2}{7}, & x = 1, 4, \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 5. Suppose X is a random variable whose pmf at $x = 0, 1, 2, 3, 4$ is given by $p_X(x) = \frac{2x+1}{25}$.

Remark: Note that the statement above does not specify the value of the $p_X(x)$ at the value of x that is not 0,1,2,3, or 4.

(a) What is $p_X(5)$?

(b) Determine the following probabilities:

(i) $P[X = 4]$

(ii) $P[X \leq 1]$

(iii) $P[2 \leq X < 4]$

(iv) $P[X > -10]$

Solution:

(a) First, we calculate

$$\sum_{x=0}^{4} p_X(x) = \sum_{x=0}^{4} \frac{2x+1}{25} = \frac{1+3+5+7+9}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other x with $p_X(x) > 0$. At $x = 5$, we then conclude that $p_X(5) = \boxed{0.}$ The same reasoning also implies that $p_X(x) = 0$ at any x that is not 0,1,2,3, or 4.

(b) Recall that, for discrete random variable X, the probability

$$P[\text{some condition(s) on } X]$$

can be calculated by adding $p_X(x)$ for all x in the support of X that satisfies the given condition(s).

(i) $P[X = 4] = p_X(4) = \frac{2\times4+1}{25} = \boxed{\frac{9}{25}.}$

(ii) $P[X \leq 1] = p_X(0) + p_X(1) = \frac{2\times0+1}{25} + \frac{2\times1+1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}.}$

(iii) $P[2 \leq X < 4] = p_X(2) + p_X(3) = \frac{2\times2+1}{25} + \frac{2\times3+1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}.}$

(iv) $P[X > -10] = \boxed{1}$ because all the x in the support of X satisfies $x > -10$.

Extra Question

Here is an optional question for those who want more practice.

Problem 6. The circuit in Figure 7.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-35]

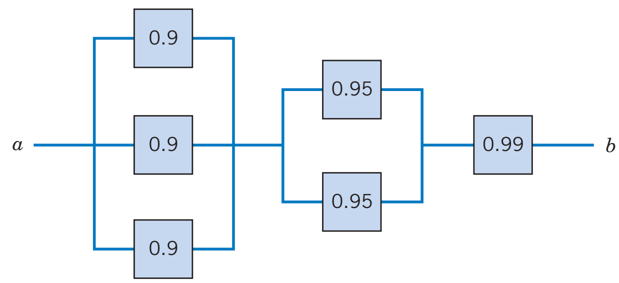

Figure 7.1: Circuit for Problem 6

Solution: The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left. From the independence and based upon Problem 5 in HW6,

$$P(L) = 1 - (1 - 0.9)^3 = 1 - 0.1^3 = 0.999.$$

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - (1 - 0.95)^2 = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Finally, the probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.

Therefore, with the independence assumption used again, along with similar reasoning to the solution of Problem 1 in HW6, the solution is

$$0.999 \times 0.9975 \times 0.99 = 0.986537475 \approx \boxed{0.987}.$$