EES 315: Probability and Random Processes	**2020/1**

HW Solution 5 — Due: Not Due

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Problem 1.

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A \cap B)$

(ii) $P(A^c \cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

Solution:

(a)

(i) By definition, $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.4 \times 0.5 = \boxed{0.2.}$$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, one can apply the property $P(A^c|B) = 1 - P(A|B)$ to get

$$P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3.$$

(b) **Method 1**: By definition, $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.2 \times 0.8 = 0.16.$$

Next, from $P(B) = 0.8$, we know that

$$P(B^c) = 1 - P(B) = 1 - 0.8 = 0.2.$$

By definition, $P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)}$. Therefore,

$$P(A \cap B^c) = P(A|B^c)P(B^c) = 0.3 \times 0.2 = 0.06.$$

Hence, $P(A) = P(A \cap B) + P(A \cap B^c) = 0.16 + 0.16 = \boxed{0.22.}$

Method 2: By the total probability formula, $P(A) = P(A|B)P(B)+P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$.

Method 3: For those who are not seeking a "smart" way to solve this question, we can try the following:

Note that when we have two events, the sample space is always partitioned into four events: $A \cap B$, $A^c \cap B$, $A \cap B^c$, and $A^c \cap B^c$. (It might be helpful to draw the Venn diagram here.) Let's define their probabilities as p_1, p_2, p_3, and p_4, respectively. We are given three conditions which can then be turned into three equations. There is also one extra condition that $p_1 + p_2 + p_3 + p_4 = 1$. Therefore, we have four equations with four unknowns. Applying some high-school algebra, we should be able to solve for p_1, p_2, p_3, and p_4. With these, we can calculate probability of any event. For example, $P(A) = p_1 + p_3$.

Problem 2. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

(a) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) What is $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

Solution:

(a) Because the test is correct 99% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.01}.$$

(b) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where $P(+)$ can be evaluated by the total probability formula:

$$P(+) = P(+|H)P(H) + P(+|H^c)P(H^c) = 0.99 \times 0.0002 + 0.01 \times 0.9998.$$

Plugging this back into the Bayes' formula gives

$$P(H|+) = \frac{0.99 \times 0.0002}{0.99 \times 0.0002 + 0.01 \times 0.9998} \approx \boxed{0.0194}.$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 3. Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4. Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped. Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped. [Gubner, 2006, Ex.1.20]

(a) Find the probability that a packet is dropped.
Hint: Use total probability theorem.

(b) Find the conditional probability that a packet is routed through El Paso given that it is not dropped.
Hint: Use Bayes' theorem.

Solution: To solve this problem, we use the notation $E = \{\text{routed through El Paso}\}$ and $D = \{\text{packet is dropped}\}$. With this notation, it is easy to interpret the problem as telling us that

$$P(D|E) = 1/3, \quad P(D|E^c) = 1/4, \quad \text{and } P(E) = 3/4.$$

(a) By the law of total probability,

$$\begin{aligned} P(D) &= P(D|E)P(E) + P(D|E^c)P(E^c) = (1/3)(3/4) + (1/4)(1 - 3/4) \\ &= 1/4 + 1/16 = \boxed{5/16} = 0.3125. \end{aligned}$$

(b) $P(E|D^c) = \frac{P(E \cap D^c)}{P(D^c)} = \frac{P(D^c|E)P(E)}{P(D^c)} = \frac{(1-1/3)(3/4)}{1-5/16} = \boxed{\frac{8}{11}} \approx 0.7273.$

Problem 4. You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely is it that it is the fair one? [Capinski and Zastawniak, 2003, Q7.28]

Solution: Let $F, U,$ and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the probability $P(F)$ of selecting the fair coin is $P(F) = \frac{1}{2}$. For fair coin, the conditional probability $P(H|F)$ of heads is $\frac{1}{2}$ For the unfair coin, $P(U) = 1 - P(F) = \frac{1}{2}$ and $P(H|U) = \frac{1}{3}$.

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3}} = \frac{1}{1 + \frac{2}{3}} = \boxed{\frac{3}{5}}.$$

Problem 5. You have three coins in your pocket, two fair ones but the third biased with probability of heads p and tails $1-p$. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins? [Capinski and Zastawniak, 2003, Q7.29]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively. We are given that

$$P(F) = \frac{2}{3}, \quad P(U) = \frac{1}{3}, \quad P(H|F) = \frac{1}{2}, P(H|U) = p.$$

By Bayes' theorem, the probability that one of the fair coins has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{1}{2} \times \frac{2}{3} + p \times \frac{1}{3}} = \boxed{\frac{1}{1+p}}.$$

Alternative Solution: Let F_1, F_2, U and H be the events that "the selected coin is the first fair coin", "the selected coin is the second fair coin", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the events $F1, F2$, and U are equally likely:

$$P(F_1) = P(F_2) = P(U) = \frac{1}{3}.$$

For fair coins, the conditional probability of heads is $\frac{1}{2}$ and for the unfair coin, the conditional probability of heads is p:

$$P(H|F_1) = P(H|F_2) = \frac{1}{2}, \quad P(H|U) = p.$$

The probability that one of the fair coins has been selected given that it lands heads up is $P(F_1 \cup F_2|H)$. Now F_1 and F_2 are disjoint events. Therefore,

$$P(F_1 \cup F_2|H) = P(F_1|H) + P(F_2|H).$$

By Bayes' theorem,

$$P(F_1|H) = \frac{P(H|F_1)P(F_1)}{P(H)} \quad \text{and} \quad P(F_2|H) = \frac{P(H|F_2)P(F_2)}{P(H)}.$$

Therefore,

$$P(F_1 \cup F_2|H) = \frac{P(H|F_1)P(F_1)}{P(H)} + \frac{P(H|F_2)P(F_2)}{P(H)} = \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H)}.$$

Note that the collection of three events F_1, F_2, and U partitions the sample space. Therefore, by the total probability theorem,

$$P(H) = P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U).$$

Plugging the above expression of $P(H)$ into our expression for $P(F_1 \cup F_2|H)$ gives

$$\begin{aligned} P(F_1 \cup F_2|H) &= \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U)} \\ &= \frac{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}+p\times\frac{1}{3}} = \boxed{\frac{1}{1+p}}. \end{aligned}$$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 6. Someone has rolled a fair dice twice. Suppose he tells you that "one of the rolls turned up a face value of six". What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Note the followings:

- The answer is not $\frac{1}{6}$.
- Although there is no use of the word "given" or "conditioned on" in this question, the probability we seek is a conditional one. We have an extra piece of information because we know that the event "one of the rolls turned up a face value of six" has occurred.
- The question says "one of the rolls" without telling us which roll (the first or the second) it is referring to.

Solution: Let the sample space be the set $\{(i,j)|i,j = 1,\dots,6\}$, where i and j denote the outcomes of the first and second rolls, respectively. They are all equally likely; so each has probability of 1/36. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = \{(1,6),\dots,(5,6),(6,6),(6,5),\dots,(6,1)\}$. Applying the definition of conditional probability gives

$$P(A|B) = P(A \cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.

Problem 7.

(a) Suppose that $P(A|B) = 1/3$ and $P(A|B^c) = 1/4$. Find the range of the possible values for $P(A)$.

(b) Suppose that C_1, C_2, and C_3 partition Ω. Furthermore, suppose we know that $P(A|C_1) = 1/3$, $P(A|C_2) = 1/4$ and $P(A|C_3) = 1/5$. Find the range of the possible values for $P(A)$.

Solution: First recall the total probability theorem: Suppose we have a collection of events $B_1, B_2, \dots, B_n$ which partitions Ω. Then,

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_n) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots P(A|B_n)P(B_n) \end{aligned}$$

(a) Note that B and B^c partition Ω. So, we can apply the total probability theorem:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)).$$

You may check that, by varying the value of $P(B)$ from 0 to 1, we can get the value of $P(A)$ to be any number in the range $\left[\frac{1}{4}, \frac{1}{3}\right]$. Technically, we can not use $P(B) = 0$ because that would make $P(A|B)$ not well-defined. Similarly, we can not use $P(B) = 1$ because that would mean $P(B^c) = 0$ and hence make $P(A|B^c)$ not well-defined. Therfore, the range of $P(A)$ is $\boxed{\left(\frac{1}{4}, \frac{1}{3}\right).}$

Note that larger value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) < \frac{1}{3}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{3}.$$

Similarly, smaller value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) > \frac{1}{4}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{4}.$$

(b) Again, we apply the total probability theorem:

$$\begin{aligned} P(A) &= P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + P(A|C_3)P(C_3) \\ &= \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3). \end{aligned}$$

Because C_1, C_2, and C_3 partition Ω, we know that $P(C_1) + P(C_2) + P(C_3) = 1$. Now,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) < \frac{1}{3}P(C_1) + \frac{1}{3}P(C_2) + \frac{1}{3}P(C_3) = \frac{1}{3}.$$

Similarly,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) > \frac{1}{5}P(C_1) + \frac{1}{5}P(C_2) + \frac{1}{5}P(C_3) = \frac{1}{5}.$$

Therefore, $P(A)$ must be inside $\left(\frac{1}{5}, \frac{1}{3}\right)$.

You may check that any value of $P(A)$ in the range $\boxed{\left(\frac{1}{5}, \frac{1}{3}\right)}$ can be obtained by first setting the value of $P(C_2)$ to be close to 0 and varying the value of $P(C_1)$ from 0 to 1.

Problem 8. In his book *Chances: Risk and Odds in Everyday Life*, James Burke says that there is a 72% chance a polygraph test (lie detector test) will catch a person who is, in fact, lying. Furthermore, there is approximately a 7% chance that the polygraph will falsely accuse someone of lying. [Brase and Brase, 2011, Q4.2.26]

(a) If the polygraph indicated that 30% of the questions were answered with lies, what would you estimate for the actual percentage of lies in the answers?

(b) If the polygraph indicated that 70% of the questions were answered with lies, what would you estimate for the actual percentage of lies?

Solution: Let AT and AL be the events that "the person actually answers the truth" and "the person actually answers with lie", respectively. Also, let PT and PL be the events that "the polygraph indicates that the answer is the truth" and "the polygraph indicates that the answer is a lie", respectively.

We know, from the provided information, that $P(PL|AL) = 0.72$ and that $P(PL|AT) = 0.07$.

Applying the total probability theorem, we have

$$\begin{aligned} P(PL) &= P(PL|AL)P(AL) + P(PL|AT)P(AT) \\ &= P(PL|AL)P(AL) + P(PL|AT)(1 - P(AL)). \end{aligned}$$

Solving for $P(AL)$, we have

$$P\left(AL\right) = \frac{P\left(PL\right) - P\left(PL\left|AT\right.\right)}{P\left(PL\left|AL\right.\right) - P\left(PL\left|AT\right.\right)} = \frac{P\left(PL\right) - 0.07}{0.72 - 0.07} = \frac{P\left(PL\right) - 0.07}{0.65}.$$

(a) Plugging in $P\left(PL\right) = 0.3$, we have $P\left(AL\right) = \boxed{0.3538}$.

(b) Plugging in $P\left(PL\right) = 0.7$, we have $P\left(AL\right) = \boxed{0.9692}$.

Problem 9. Software to detect fraud in consumer phone cards tracks the number of metropolitan areas where calls originate each day. It is found that 1% of the legitimate users originate calls from two or more metropolitan areas in a single day. However, 30% of fraudulent users originate calls from two or more metropolitan areas in a single day. The proportion of fraudulent users is 0.01%. If the same user originates calls from two or more metropolitan areas in a single day, what is the probability that the user is fraudulent? [Montgomery and Runger, 2010, Q2-144]

Solution: Let F denote the event of fraudulent user and let M denote the event of originating calls from multiple (two or more) metropolitan areas in a day. Then,

$$P(F|M) = \frac{P(M|F)P(F)}{P(M|F)P(F) + P(M|F^c)P(F^c)} = \frac{1}{1 + \frac{P(M|F^c)}{P(M|F)} \times \frac{P(F^c)}{P(F)}}$$
$$= \frac{1}{1 + \frac{\frac{1}{100}}{\frac{30}{100}} \times \frac{\frac{9999}{10^4}}{\frac{1}{10^4}}} = \frac{1}{1 + \frac{9999}{30}} = \frac{30}{30 + 9999} = \frac{30}{10029} \approx \boxed{0.0030}.$$

Problem 10. An article in the British Medical Journal ["Comparison of Treatment of Renal Calculi by Operative Surgery, Percutaneous Nephrolithotomy, and Extracorporeal Shock Wave Lithotripsy" (1986, Vol. 82, pp. 879892)] provided the following discussion of success rates in kidney stone removals. Open surgery (OS) had a success rate of 78% (273/350) while a newer method, percutaneous nephrolithotomy (PN), had a success rate of 83% (289/350). This newer method looked better, but the results changed when stone diameter was considered. For stones with diameters less than two centimeters, 93% (81/87) of cases of open surgery were successful compared with only 87% (234/270) of cases of PN. For stones greater than or equal to two centimeters, the success rates were 73% (192/263) and 69% (55/80) for open surgery and PN, respectively. Open surgery is better for both stone sizes, but less successful in total. In 1951, E. H. Simpson pointed out this apparent contradiction (known as Simpson's Paradox) but the hazard still persists today. Explain how open surgery can be better for both stone sizes but worse in total. [Montgomery and Runger, 2010, Q2-115]

Solution: First, let's recall the total probability theorem:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)P(B) + P(A|B^c)P(B^c). \end{aligned}$$

We can see that $P(A)$ does not depend only on $P(A|B)$ and $P(A|B^c)$. It also depends on $P(B)$ and $P(B^c)$. In the extreme case, we may imagine the case with $P(B) = 1$ in which $P(A) = P(A|B)$. At another extreme, we may imagine the case with $P(B) = 0$ in which $P(A) = P(A|B^c)$. Therefore, depending on the value of $P(B)$, the value of $P(A)$ can be anywhere between $P(A|B)$ and $P(A|B^c)$.

Now, let's consider events A_1, B_1, A_2, and B_2. Let $P(A_1|B_1) = 0.93$ and $P(A_1|B_1^c) = 0.73$. Therefore, $P(A_1) \in [0.73, 0.93]$. On the other hand, let $P(A_2|B_2) = 0.87$ and $P(A_2|B_2^c) = 0.69$. Therefore, $P(A_2) \in [0.69, 0.87]$. With small value of $P(B_1)$, the value of $P(A_1)$ can be 0.78 which is closer to its lower end of the bound. With large value of $P(B_2)$, the value of $P(A_2)$ can be 0.83 which is closer to its upper end of the bound. Therefore, even though $P(A_1|B_1) > P(A_2|B_2) = 0.87$ and $P(A_1|B_1^c) > P(A_2|B_2^c)$, it is possible that $P(A_1) < P(A_2)$.

In the context of the paradox under consideration, note that the success rate of PN with small stones (87%) is higher than the success rate of OS with large stones (73%). Therefore, by having a lot of large stone cases to be tested under OS and also have a lot of small stone cases to be tested under PN, we can create a situation where the overall success rate of PN comes out to be better then the success rate of OS. This is exactly what happened in the study as shown in Table 5.1.

Open surgery

	success	failure	sample size	sample percentage	conditional success rate
large stone	192	71	263	75%	73%
small stone	81	6	87	25%	93%
overall summary	273	77	350	100%	78%

PN

	success	failure	sample size	sample percentage	conditional success rate
large stone	55	25	80	23%	69%
small stone	234	36	270	77%	87%
overall summary	289	61	350	100%	83%

Table 5.1: Success rates in kidney stone removals.