EES 315: Probability and Random Processes **2020/1**

HW Solution 3 — Due: September 16, 11:59 PM

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Problem 1. (**Classical Probability and Combinatorics**) A bin of 50 parts contains five that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective. [Montgomery and Runger, 2010, Q2-49]

Solution: The number of ways to select two parts from 50 is $\binom{50}{2}$ and the number of ways to select two defective parts from the 5 defective ones is $\binom{5}{2}$ Therefore the probability is

$$\frac{\binom{5}{2}}{\binom{50}{2}} = \frac{2}{245} = \boxed{0.0082}.$$

Alternatively, if the two parts in the sample are selected one by one, then we may also consider their ordering as well. In such case, we use the formula for "ordered sampling without replacement" instead of "unordered sampling without replacement":

$$\frac{(5)_2}{(50)_2} = \frac{5 \times 4}{50 \times 49} = \frac{2}{245} = \boxed{0.0082}.$$

Problem 2. (**Classical Probability and Combinatorics**) We all know that the chance of a head (H) or tail (T) coming down after a fair coin is tossed are fifty-fifty. If a fair coin is tossed ten times, then intuition says that five heads are likely to turn up.

Calculate the probability of getting exactly five heads (and hence exactly five tails).

Solution: There are 2^{10} possible outcomes for ten coin tosses. (For each toss, there is two possibilities, H or T). Only $\binom{10}{5}$ among these outcomes have exactly heads and five tails. (Choose 5 positions from 10 position for H. Then, the rest of the positions are automatically T.) The probability of have exactly 5 H and 5 T is

$$\boxed{\frac{\binom{10}{5}}{2^{10}} \approx 0.246.}$$

Note that five heads and five tails will turn up more frequently than any other single combination (one head, nine tails for example) but the sum of all the other possibilities is much greater than the single 5 H, 5 T combination.

Problem 3. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

(c) Use the binomial theorem (3.5) to evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}$.

Solution:

(a) The coefficient of $x^r y^{n-r}$ is $\binom{n}{r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12} = \boxed{5,200,300}$.

(b) We start from the expansion of $(a+b)^n$. Then we set $a = 2x$ and $b = -3y$:

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r} = \sum_{r=0}^{n}\binom{n}{r}(2x)^r(-3y)^{n-r} = .\sum_{r=0}^{n}\binom{n}{r}2^r(-3)^{n-r}x^r y^{n-r}. \tag{3.2}$$

Therefore, the coefficient of $x^r y^{n-r}$ is $\binom{n}{r}2^r(-3)^{n-r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12}2^{12}(-3)^{13} = -\frac{25!}{12!13!}2^{12}3^{13} = \boxed{-33959763545702400}$.

(c) From (3.5), set $x = -1$ and $y = 1$, then we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (-1+1)^n = \boxed{0}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 4. *An Even Split at Coin Tossing*: Let p_n be the probability of getting exactly n heads (and hence exactly n tails) when a fair coin is tossed $2n$ times.

(a) Find p_n.

(b) Sometimes, to work theoretically with large factorials, we use Stirling's Formula:

$$n! \approx \sqrt{2\pi n}n^n e^{-n} = \left(\sqrt{2\pi e}\right)e^{\left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)}. \tag{3.3}$$

Approximate p_n using Stirling's Formula.

(c) Find $\lim_{n\to\infty} p_n$.

Solution: Note that we have worked on a particular case ($n = 5$) of this problem earlier.

(a) Use the same solution as Problem 2; change 5 to n and 10 to $2n$, we have

$$p_n = \boxed{\frac{\binom{2n}{n}}{2^{2n}}}.$$

(b) By Stirling's Formula, we have

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \approx \frac{\sqrt{2\pi 2n}(2n)^{2n}e^{-2n}}{\left(\sqrt{2\pi n}n^n e^{-n}\right)^2} = \frac{4^n}{\sqrt{\pi n}}.$$

Hence,

$$p_n \approx \boxed{\frac{1}{\sqrt{\pi n}}}. \tag{3.4}$$

[Mosteller, *Fifty Challenging Problems in Probability with Solutions*, 1987, Problem 18]
See Figure 3.1 for comparison of p_n and its approximation via Stirling's formula.

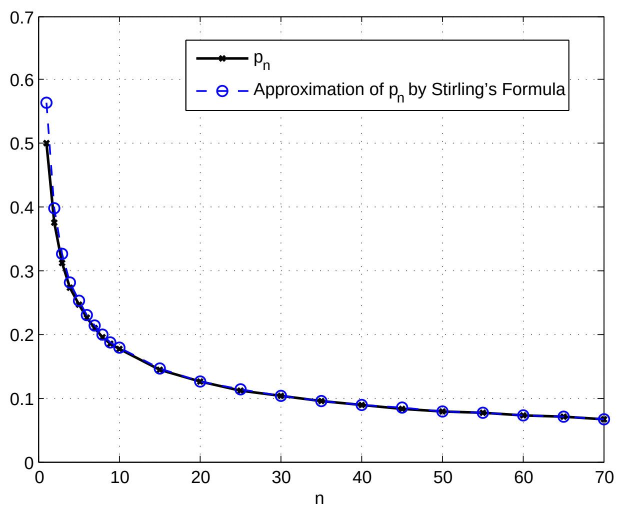

Figure 3.1: Comparison of p_n and its approximation via Stirling's formula

(c) From (3.4), $\lim_{n\to\infty} p_n = \boxed{0}$. A more rigorous proof of this limit would use the bounds

$$\frac{4^n}{\sqrt{4n}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{3n+1}}.$$

Problem 5. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.5}$$

(a) Use the binomial theorem (3.5) to simplify the following sums

(i) $\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(ii) $\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(b) If we differentiate (3.5) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r \binom{n}{r} x^r y^{n-r} = nx(x+y)^{n-1}.$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r}$.

Solution:

(a) To deal with the sum involving only the even terms (or only the odd terms), we first use (3.5) to expand $(x+y)^n$ and $(x+(-y))^n$. When we add the expanded results, only the even terms in the sum are left. Similarly, when we find the difference between the two expanded results, only the the odd terms are left. More specifically,

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n + (y-x)^n\right), \text{ and}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n - (y-x)^n\right).$$

If $x+y=1$, then

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 + (1-2x)^n\right)}, \text{ and} \tag{3.6a}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 - (1-2x)^n\right)}. \tag{3.6b}$$

(b) $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r} = \boxed{nx\left(x(n-1)(x+y)^{n-2} + (x+y)^{n-1}\right)}$.