EES 315: Probability and Random Processes **2020/1**

HW Solution 10 — Due: Not Due

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Problem 1 (Yates and Goodman, 2005, Q3.2.1)**.** The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Use the pdf to find the following quantities.

(a) the unknown constant c

(b) $P[0 \le X \le 1]$

(c) $P[-1/2 \le X \le 1/2]$.

Solution:

(a) Recall that any pdf should integrate to 1. Here,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 cx\,dx = c\left.\frac{x^2}{2}\right|_0^2 = 2c.$$

Equating the expression above to 1, we get $c = \boxed{\frac{1}{2}}$.

(b) $P[0 \le X \le 1] = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^1 = \boxed{\frac{1}{4}}$.

(c) $P\left[-\frac{1}{2} \le X \le \frac{1}{2}\right] = \int_{-1/2}^{1/2} f_X(x)\,dx$. Now, $f_X(x) = 0$ on the interval $\left[-\frac{1}{2}, 0\right)$. Therefore, we don't have to integrate over that interval and

$$P\left[-\frac{1}{2} \le X \le \frac{1}{2}\right] = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^{1/2} = \boxed{\frac{1}{16}}.$$

Problem 2 (Yates and Goodman, 2005, Q3.3.4)**.** The pdf of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[Y]$.

(b) Find $\operatorname{Var} Y$.

Solution:

(a) Recall that, for continuous random variable Y,

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Note that when y is outside of the interval $[0,2)$, $f_Y(y) = 0$ and hence does not affect the integration. We only need to integrate over $[0,2)$ in which $f_Y(y) = \frac{y}{2}$. Therefore,

$$\mathbb{E}Y = \int_0^2 y\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^2}{2}dy = \left.\frac{y^3}{2\times 3}\right|_0^2 = \boxed{\frac{4}{3}}.$$

(b) The variance of any random variable Y (discrete or continuous) can be found from

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2$$

We have already calculate $\mathbb{E}Y$ in the previous part. So, now we need to calculate $\mathbb{E}[Y^2]$. Recall that, for continuous random variable,

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy.$$

Here, $g(y) = y^2$. Therefore,

$$\mathbb{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy.$$

Again, in the integration, we can ignore the y whose $f_Y(y) = 0$:

$$\mathbb{E}\left[Y^2\right] = \int_0^2 y^2\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^3}{2}dy = \left.\frac{y^4}{2\times 4}\right|_0^2 = \boxed{2}.$$

Plugging this into the variance formula gives

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}.$$

Problem 3. The pdf of random variable V is

$$f_V(v) = \begin{cases} \frac{v+5}{72}, & -5 < v < 7, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is $\mathbb{E}[V]$?

(b) What is $\text{Var}[V]$?

(c) What is $\mathbb{E}[V^3]$?

Solution:

(a) $\mathbb{E}[V] = \int_{-\infty}^{\infty} v f_V(v)\, dv = \int_{-5}^{7} v \left(\frac{v+5}{72}\right) dv = \frac{1}{72} \int_{-5}^{7} v^2 + 5v dv = \boxed{3}$.

(b) $\mathbb{E}[V^2] = \int_{-\infty}^{\infty} v^2 f_V(v)\, dv = \int_{-5}^{7} v^2 \left(\frac{v+5}{72}\right) dv = 17.$

Therefore, $\text{Var}\, V = \mathbb{E}[V^2] - (\mathbb{E}[V])^2 = 17 - 9 = \boxed{8}$.

(c) $\mathbb{E}[V^3] = \int_{-\infty}^{\infty} v^3 f_V(v)\, dv = \int_{-5}^{7} v^3 \left(\frac{v+5}{72}\right) dv = \boxed{\frac{431}{5} = 86.2}$.

Problem 4 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

(b) What is $\mathbb{E}[X]$?

(c) What is $\mathbb{E}[X^5]$?

(d) What is $\mathbb{E}\left[e^X\right]$?

Solution: For a uniform random variable X on the interval (a, b), we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

In this problem, we have $a = -5$ and $b = 5$.

(a) $f_X(x) = \boxed{\begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \le x \le 5 \end{cases}}$

(b) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\,dx = \int\limits_{-5}^{5} x \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^2}{2}\right|_{-5}^{5} = \frac{1}{20}\left(5^2 - (-5)^2\right) = \boxed{0}$.

In general,

$$\mathbb{E}X = \int\limits_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}X = \boxed{0}$.

(c) $\mathbb{E}\left[X^5\right] = \int\limits_{-\infty}^{\infty} x^5 f_X(x)\,dx = \int\limits_{-5}^{5} x^5 \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^6}{6}\right|_{-5}^{5} = \frac{1}{60}\left(5^6 - (-5)^6\right) = \boxed{0}$.

In general,

$$\mathbb{E}\left[X^5\right] = \int\limits_a^b x^5 \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x^5 dx = \frac{1}{b-a} \left.\frac{x^6}{6}\right|_a^b = \frac{1}{b-a} \frac{b^6 - a^6}{6}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[X^5\right] = \boxed{0}$.

(d) In general,

$$\mathbb{E}\left[e^X\right] = \int\limits_a^b e^x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b e^x dx = \frac{1}{b-a} \left. e^x \right|_a^b = \frac{e^b - e^a}{b-a}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[e^X\right] = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84$.

Problem 5 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi} 1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}\left[g(\Theta)\right] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}\left[X\right] = 5\mathbb{E}\left[\cos(7t+\Theta)\right] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t+\theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}\left[X\right] = \boxed{0}$.

(b) Y is another function of Θ.

$$\begin{aligned}\mathbb{E}\left[Y\right] &= \mathbb{E}\left[5\cos(7t_1+\Theta)\times 5\cos(7t_2+\Theta)\right] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1+\theta)\times 5\cos(7t_2+\theta)d\theta \\ &= \frac{25}{2\pi}\int_0^{2\pi} \cos(7t_1+\theta)\times\cos(7t_2+\theta)d\theta.\end{aligned}$$

Recall[1] the cosine identity

$$\cos(a)\times\cos(b) = \frac{1}{2}\left(\cos\left(a+b\right)+\cos\left(a-b\right)\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}Y &= \frac{25}{4\pi}\int_0^{2\pi} \cos\left(7t_1+7t_2+2\theta\right)+\cos\left(7\left(t_1-t_2\right)\right)d\theta \\ &= \frac{25}{4\pi}\left(\int_0^{2\pi}\cos\left(7t_1+7t_2+2\theta\right)d\theta + \int_0^{2\pi}\cos\left(7\left(t_1-t_2\right)\right)d\theta\right).\end{aligned}$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos\left(7\left(t_1-t_2\right)\right)\int_0^{2\pi} d\theta = \frac{25}{4\pi}\cos\left(7\left(t_1-t_2\right)\right)2\pi = \boxed{\frac{25}{2}\cos\left(7\left(t_1-t_2\right)\right)}.$$

[1] This identity could be derived easily via the Euler's identity:

$$\begin{aligned}\cos(a)\times\cos(b) &= \frac{e^{ja}+e^{-ja}}{2}\times\frac{e^{jb}+e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb}+e^{-ja}e^{jb}+e^{ja}e^{-jb}+e^{-ja}e^{-jb}\right) \\ &= \frac{1}{2}\left(\frac{e^{ja}e^{jb}+e^{-ja}e^{-jb}}{2}+\frac{e^{-ja}e^{jb}+e^{ja}e^{-jb}}{2}\right) \\ &= \frac{1}{2}\left(\cos\left(a+b\right)+\cos\left(a-b\right)\right).\end{aligned}$$

Problem 6. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

Solution: Let T be the time to failure (in hours). We are given that $T \sim \mathcal{E}(\lambda)$ where $\lambda = 3 \times 10^{-4}$. Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find $P\left[T > 10^4\right]$.

We shall first provide the general formula for the ccdf $P[T > t]$ when $t > 0$:

$$P[T > t] = \int_t^\infty f_T(\tau) d\tau = \int_t^\infty \lambda e^{-\lambda\tau} d\tau = -e^{-\lambda\tau}\Big|_t^\infty = e^{-\lambda t}. \tag{10.1}$$

Therefore,

$$P\left[T > 10^4\right] = e^{-3\times10^{-4}\times10^4} = \boxed{e^{-3} \approx 0.0498.}$$

(b) We start with $P[T \le 7000] = 1 - P[T > 7000]$. Next, we apply (10.1) to get

$$P[T \le 7000] = 1 - P[T > 7000] = 1 - e^{-3\times10^{-4}\times7000} = \boxed{1 - e^{-2.1} \approx 0.8775.}$$

Problem 7. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find the expected value of X.

(c) Find the variance and the standard deviation of X.

(d) Find the expected value of power when the resistance is 100 ohms?

Solution:

(a) $P[X < 5] = \int\limits_{-\infty}^{5} f_X(x)dx = \int\limits_{-\infty}^{0} \underbrace{f_X(x)}_{0}dx + \int\limits_{0}^{5} \underbrace{f_X(x)}_{5}dx = 0 + 5x\Big|_{x=4.9}^{5} = \boxed{0.5}.$

(b) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\,dx = \int\limits_{-\infty}^{4.9} x \underbrace{f_X(x)}_{0} dx + \int\limits_{4.9}^{5.1} x \underbrace{f_X(x)}_{5} dx + \int\limits_{5.1}^{x} x \underbrace{f_X(x)}_{0} dx = 0 + 5\frac{x^2}{2}\Big|_{x=4.9}^{5.1} + 0 = \boxed{5}$ mA.

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\mathbb{E}X = \frac{a+b}{2} = \frac{4.9+5.1}{2} = 5$.

(c) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$. From the previous part, we know that $\mathbb{E}X = 5$. SO, to find $\operatorname{Var} X$, we need to find $\mathbb{E}[X^2]$: $\mathbb{E}[X^2] = \int\limits_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int\limits_{-\infty}^{4.9} x^2 \underbrace{f_X(x)}_{0} dx + \int\limits_{4.9}^{5.1} x^2 \underbrace{f_X(x)}_{5} dx + \int\limits_{5.1}^{x} x^2 \underbrace{f_X(x)}_{0} dx = 0 + 5\frac{x^3}{3}\Big|_{x=4.9}^{5.1} + 0 = 25 + \frac{1}{300}$.

Therefore, $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \left(25 + \frac{1}{300}\right) = \boxed{\frac{1}{300}} \approx 0.0033\ (\text{mA})^2$

and

$\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\frac{1}{10\sqrt{3}}} \approx 0.0577$ mA.

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\operatorname{Var} X = \frac{(b-a)^2}{12} = \frac{(5.1-4.9)^2}{12} = \frac{1}{300}$.

(d) Recall that $P = I \times V = I^2 r$. Here, $I = X$. Therefore, $P = X^2 r$ and $\mathbb{E}P = \mathbb{E}[X^2 r] = r\mathbb{E}[X^2] = 100 \times \left(25 + \frac{1}{300}\right) = 2500 + \frac{1}{3} \approx 2.50033 \times 10^3\ [(\text{mA})^2\Omega]$. Factoring out m^2, we have $\mathbb{E}P \approx 2.50033$ mW. ($[\text{A}^2\Omega] = [\text{W}]$.)

Problem 8. Let X be a uniform random variable on the interval $[0,1]$. Set

$$A = \left[0, \frac{1}{2}\right), \quad B = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), \quad \text{and } C = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \cup \left[\frac{3}{4}, \frac{7}{8}\right).$$

Are the events $[X \in A], [X \in B]$, and $[X \in C]$ independent?

Solution: Note that

$$P[X \in A] = \int_0^{\frac{1}{2}} dx = \frac{1}{2},$$

$$P[X \in B] = \int_0^{\frac{1}{4}} dx + \int_{\frac{1}{2}}^{\frac{3}{4}} dx = \frac{1}{2}, \text{ and}$$

$$P[X \in C] = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx + \int_{\frac{3}{4}}^{\frac{7}{8}} dx = \frac{1}{2}.$$

Now, for pairs of events, we have

$$P([X \in A] \cap [X \in B]) = \int_0^{\frac{1}{4}} dx = \frac{1}{4} = P[X \in A] \times P[X \in B], \tag{10.2}$$

$$P([X \in A] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx = \frac{1}{4} = P[X \in A] \times P[X \in C], \text{ and} \tag{10.3}$$

$$P([X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx = \frac{1}{4} = P[X \in B] \times P[X \in C]. \tag{10.4}$$

Finally,

$$P([X \in A] \cap [X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx = \frac{1}{8} = P[X \in A]\, P[X \in B]\, P[X \in C]. \tag{10.5}$$

From (10.2), (10.3), (10.4) and (10.5), we can conclude that the events $[X \in A], [X \in B]$, and $[X \in C]$ are $\boxed{\text{independent}}$.