EES 315: Probability and Random Processes **2020/1**

HW Solution 6 — Due: October 22, 11:59 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Series Circuit: The circuit in Figure 6.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-32]

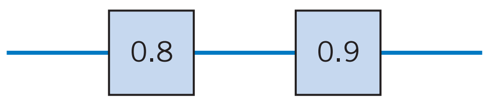

Figure 6.1: Circuit for Problem 1

Solution: Let L and R denote the events that the left and right devices operate, respectively. For a path to exist, both need to operate. Therefore, the probability that the circuit operates is $P(L \cap R)$.

We are told that $L^c \perp\!\!\!\perp R^c$. This is equivalent to $L \perp\!\!\!\perp R$. By their independence,

$$P(L \cap R) = P(L)P(R) = 0.8 \times 0.9 = \boxed{0.72}.$$

Problem 2. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) **Method 1**: $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) **Method 1**: First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) **Method 1**: $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.

Method 2: Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Problem 3. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1,2,3,4\}$, $A_1 = \{1,2\}$, $A_2 = \{1,3\}$, $A_3 = \{2,3\}$.

(i) Determine whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

(ii) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(iii) Are A_1, A_2, and A_3 independent?

(b) Let $\Omega = \{1,2,3,4,5,6\}$, $A_1 = \{1,2,3,4\}$, $A_2 = A_3 = \{4,5,6\}$.

(i) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(ii) Check whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

(iii) Are A_1, A_2, and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

(i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P(A_1 \cap A_2 \cap A_3) = 0$, which is *not* the same as $P(A_1)P(A_2)P(A_3)$.

(iii) No. Although the three conditions for pairwise independence are satisfied, the last (product) condition for independence among three events is not.

Remark: This counter-example shows that pairwise independence does not imply independence.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

(i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.
$P(A_1)P(A_2)P(A_3) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6}$.
Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
$P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
$P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
Hence, $P(A_i \cap A_j) \neq P(A_i) P(A_j)$ for all $i \neq j$.

(iii) No. TO be independent, the three events must satisfy four conditions. Here, only one is satisfied.

Remark: This counter-example shows that one product condition does not imply independence.

Problem 4. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

(a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.

(b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.

(i) $P(A \cup B)$

(ii) $P(A \cup B^c)$

(iii) $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]

Solution:

(a) By multiplication rule, there are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \tag{6.1}$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the "4" in (6.1) is replaced by "1" because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}.}$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the "5" in the middle of (6.1) is replaced by "4" because we can't use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}.}$$

For the event $A \cap B$, we replace "4" in (6.1) by "1" because we need red color and we replace "5" in the middle of (6.1) by "4" because we can't use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$

(ii) **Method 1**: $P(A \cup B^c) = 1 - P\left((A \cup B^c)^c\right) = 1 - P(A^c \cap B)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$. Hence, $P(A^c \cup B^c) = 1 - P(A^c)P(B) = 1 - \frac{3}{4}\frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$

Method 2: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = B^c \cup (A \cap B)$. Therefore,

$$P(A \cup B^c) = P(B^c) + P(A \cap B) = 1 - P(B) + P(A)P(B) = 1 - \frac{4}{5} + \frac{1}{4}\frac{4}{5} = \frac{2}{5}.$$

Method 3: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = A \cup (A^c \cap B^c)$. Therefore, $P(A \cup B^c) = P(A) + P(A^c \cap B^c)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B^c$. Hence,

$$P(A \cup B^c) = P(A) + P(A^c)P(B^c) = P(A) + (1 - P(A))(1 - P(B)) = \frac{1}{4} + \frac{3}{4}\frac{1}{5} = \frac{2}{5}.$$

(iii) **Method 1**: $P(A^c \cup B^c) = 1 - P\left((A^c \cup B^c)^c\right) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$

Method 2: From the Venn diagram, note that $A^c \cup B^c$ can be expressed as a disjoint union: $A^c \cup B^c = (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c)$. Therefore,

$$P\left(A^c \cup B^c\right) = P\left(A^c \cap B\right) + P\left(A \cap B^c\right) + P\left(A^c \cap B^c\right).$$

Now, because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, and $A^c \perp\!\!\!\perp B^c$. Hence,

$$\begin{aligned} P\left(A^c \cup B^c\right) &= P\left(A^c\right) P\left(B\right) + P\left(A\right) P\left(B^c\right) + P\left(A^c\right) P\left(B^c\right) \\ &= \left(1 - P\left(A\right)\right) P\left(B\right) + P\left(A\right) \left(1 - P\left(B\right)\right) + \left(1 - P\left(A\right)\right)\left(1 - P\left(B\right)\right) \\ &= \frac{3}{4} \times \frac{4}{5} + \frac{1}{4} \times \frac{1}{5} + \frac{3}{4} \times \frac{1}{5} = \frac{16}{20} = \frac{4}{5} \end{aligned}$$

Problem 5. The circuit in Figure 6.2 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-34]

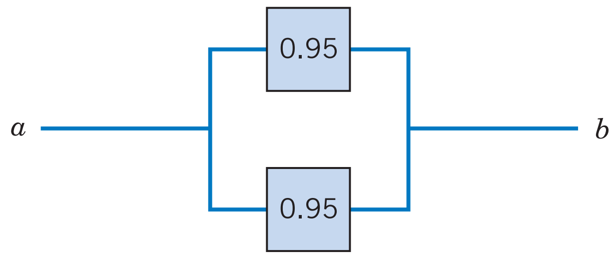

Figure 6.2: Circuit for Problem 5

Solution: Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. Therefore, the probability that the circuit operates is $P(T \cup B)$. Note that

$$P\left(T \cup B\right) = 1 - P\left((T \cup B)^c\right) = 1 - P\left(T^c \cap B^c\right).$$

We are told that $T^c \perp\!\!\!\perp B^c$. By their independence,

$$P(T^c \cap B^c) = P(T^c)P(B^c) = (1 - 0.95) \times (1 - 0.95) = 0.05^2 = 0.0025.$$

Therefore,

$$P\left(T \cup B\right) = 1 - P\left(T^c \cap B^c\right) = 1 - 0.0025 = \boxed{0.9975}.$$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 6. Show that if A and B are independent events, then so are A and B^c, A^c and B, and A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that

$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$

Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence

$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have

$$P(A^c \cap B) = P(B \cap A^c) = P(B)\,P(A^c).$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B),$$

where, for the last equality, we use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which is discussed in class.

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c, it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 7. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Hint: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. Observe that the question asks you to evaluate $P(A|(A \cup B))$.

Solution: From the question, we know that A and B are independent. The event "at least one of the two women catches nothing" can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2-p}}.$$

EES 315: Probability and Random Processes	**2020/1**

HW Solution 7 — Due: November 6, 11:59 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1 (Majority Voting in Digital Communication)**.** A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a "codeword" 111 is transmitted, and to send the message 0, a "codeword" 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Let $\mathcal{E}$ be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event $\mathcal{E}$ occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 = p^2(3-2p).$$

When $p = 0.1$, we have $P(\mathcal{E}) \approx \boxed{0.028}$.

Problem 2. For each description of a random variable X below, indicate whether X is a **discrete** random variable.

(a) X is the number of websites visited by a randomly chosen software engineer in a day.

(b) X is the number of classes a randomly chosen student is taking.

(c) X is the average height of the passengers on a randomly chosen bus.

(d) A game involves a circular spinner with eight sections labeled with numbers. X is the amount of time the spinner spins before coming to a rest.

(e) X is the thickness of the longest book in a randomly chosen library.

(f) X is the number of keys on a randomly chosen keyboard.

(g) X is the length of a randomly chosen person's arm.

Solution:We consider the number of possibilities for the values of X in each part. If the collection of possible values is countable (finite or countably infinite), then we conclude that the random variable is discrete. Otherwise, the random variable is not discrete. Therefore, the X defined in parts (a), (b), and (f) are discrete. The X defined in other parts are not discrete.

Problem 3 (Quiz4, 2014)**.** Consider a random experiment in which you roll a 20-sided fair dice. We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega, \quad Y(\omega) = (\omega - 5)^2, \quad Z(\omega) = |\omega - 5| - 3$$

Evaluate the following probabilities:

(a) $P[X = 5]$

(b) $P[Y = 16]$

(c) $P[Y > 10]$

(d) $P[Z > 10]$

(e) $P[5 < Z < 10]$

Solution: In this question, $\Omega = \{1, 2, 3, \ldots, 20\}$ because the dice has 20 sides. All twenty outcomes are equally-likely because the dice is fair. So, the probability of each outcome is $\frac{1}{20}$:

$$P(\{\omega\}) = \frac{1}{20} \text{ for any } \omega \in \Omega.$$

(a) From $X(\omega) = \omega$, we have $X(\omega) = 5$ if and only if $\omega = 5$.
Therefore, $P[X = 5] = P(\{5\}) = \boxed{\frac{1}{20}}$.

(b) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) = 16$ if and only if $\omega = \pm 4 + 5 = 1$ or 9.
Therefore, $P[Y = 16] = P(\{1, 9\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(c) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) > 10$ if and only if $(\omega - 5)^2 > 10$.
To check this, it may be more straight-forward to calculate $Y(\omega)$ at all possible values of ω:

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Y(\omega)$	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

From the table, the values of ω that satisfy the condition $Y(\omega) > 10$ are $1, 9, 10, 11, \ldots, 20$.
Therefore, $P[Y > 10] = P(\{1, 9, 10, 11, \ldots, 20\}) = \boxed{\frac{13}{20}}$.

(d) The values of ω that satisfy $|\omega - 5| - 3 > 10$ are 19 and 20.
To see this, it is straight-forward to calculate $Z(\omega)$ at all possible values of ω:

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Z(\omega)$	1	0	-1	-2	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12

Therefore, $P[Z > 10] = P(\{19, 20\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(e) The values of ω that satisfy $5 < |\omega - 5| - 3 < 10$ are $14, 15, 16, 17$.
Therefore, $P[5 < Z < 10] = P(\{14, 15, 16, 17\}) = \frac{4}{20} = \boxed{\frac{1}{5}}$.

Problem 4. Consider the sample space $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$. Suppose that $P(A) = |A|/|\Omega|$ for any event $A \subset \Omega$. Define the random variable $X(\omega) = \omega^2$. Find the probability mass function of X.

Solution: The random variable maps the outcomes $\omega = -2, -1, 0, 1, 2, 3, 4$ to numbers $x = 4, 1, 0, 1, 4, 9, 16$, respectively. Therefore,

$$
\begin{aligned}
p_X(0) &= P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7}, \\
p_X(1) &= P(\{\omega : X(\omega) = 1\}) = P(\{-1, 1\}) = \frac{2}{7}, \\
p_X(4) &= P(\{\omega : X(\omega) = 4\}) = P(\{-2, 2\}) = \frac{2}{7}, \\
p_X(9) &= P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and} \\
p_X(16) &= P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}.
\end{aligned}
$$

Combining the results above, we get the complete pmf:

$$p_X(x) = \boxed{\begin{cases} \frac{1}{7}, & x = 0, 9, 16, \\ \frac{2}{7}, & x = 1, 4, \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 5. Suppose X is a random variable whose pmf at $x = 0, 1, 2, 3, 4$ is given by $p_X(x) = \frac{2x+1}{25}$.

Remark: Note that the statement above does not specify the value of the $p_X(x)$ at the value of x that is not 0,1,2,3, or 4.

(a) What is $p_X(5)$?

(b) Determine the following probabilities:

(i) $P[X = 4]$

(ii) $P[X \leq 1]$

(iii) $P[2 \leq X < 4]$

(iv) $P[X > -10]$

Solution:

(a) First, we calculate

$$\sum_{x=0}^{4} p_X(x) = \sum_{x=0}^{4} \frac{2x+1}{25} = \frac{1+3+5+7+9}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other x with $p_X(x) > 0$. At $x = 5$, we then conclude that $p_X(5) = \boxed{0.}$ The same reasoning also implies that $p_X(x) = 0$ at any x that is not 0,1,2,3, or 4.

(b) Recall that, for discrete random variable X, the probability

$$P[\text{some condition(s) on } X]$$

can be calculated by adding $p_X(x)$ for all x in the support of X that satisfies the given condition(s).

(i) $P[X = 4] = p_X(4) = \frac{2\times4+1}{25} = \boxed{\frac{9}{25}.}$

(ii) $P[X \leq 1] = p_X(0) + p_X(1) = \frac{2\times0+1}{25} + \frac{2\times1+1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}.}$

(iii) $P[2 \leq X < 4] = p_X(2) + p_X(3) = \frac{2\times2+1}{25} + \frac{2\times3+1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}.}$

(iv) $P[X > -10] = \boxed{1}$ because all the x in the support of X satisfies $x > -10$.

Extra Question

Here is an optional question for those who want more practice.

Problem 6. The circuit in Figure 7.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-35]

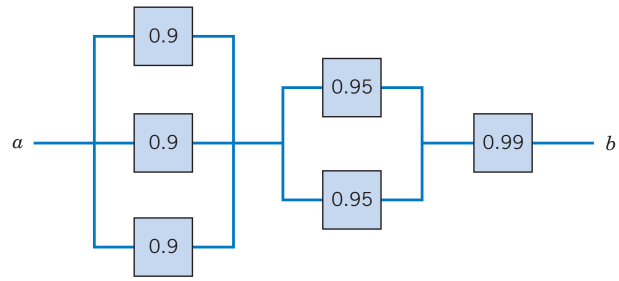

Figure 7.1: Circuit for Problem 6

Solution: The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left. From the independence and based upon Problem 5 in HW6,

$$P(L) = 1 - (1 - 0.9)^3 = 1 - 0.1^3 = 0.999.$$

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - (1 - 0.95)^2 = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Finally, the probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.

Therefore, with the independence assumption used again, along with similar reasoning to the solution of Problem 1 in HW6, the solution is

$$0.999 \times 0.9975 \times 0.99 = 0.986537475 \approx \boxed{0.987}.$$

EES 315: Probability and Random Processes **2020/1**

HW Solution 8 — Due: November 13, 11:59 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. The random variable V has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

(e) Sketch $p_V(v)$.

(f) Sketch $F_V(v)$. (Note that $F_V(v) = P[V \le v]$.)

Solution: [Y&G, Q2.2.3]

(a) We choose c so that the pmf sums to one:

$$\sum_v p_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1.$$

Hence, $c = \boxed{1/30}$.

(b) $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}] = P[V \in \{1, 4, 9, 16, 25\}] = p_V(1) + p_V(4) = c(1^2 + 4^2) = \boxed{17/30}$.

(c) $P[V \text{ even}] = p_V(2) + p_V(4) = c(2^2 + 4^2) = 20/30 = \boxed{2/3}$.

(d) $P[V > 2] = p_V(3) + p_V(4) = c(3^2 + 4^2) = 25/30 = \boxed{5/6}$.

(e) See Figure 8.1 for the sketch of $p_V(v)$:

(f) See Figure 8.2 for the sketch of $F_V(v)$:

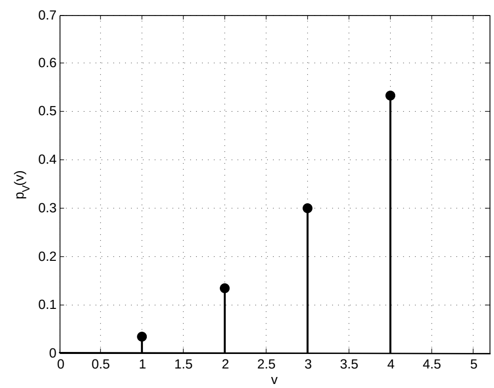

Figure 8.1: Sketch of $p_V(v)$ for Question 1

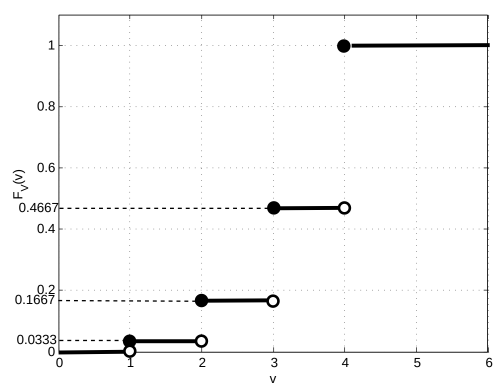

Figure 8.2: Sketch of $F_V(v)$ for Question 1

Problem 2. The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8}, \\ 0.2, & \frac{1}{8} \leq x < \frac{1}{4}, \\ 0.9, & \frac{1}{4} \leq x < \frac{3}{8}, \\ 1 & x \geq \frac{3}{8}. \end{cases}$$

Determine the following probabilities:

(a) $P[X \leq 1/18]$

(b) $P[X \leq 1/4]$

(c) $P[X \leq 5/16]$

(d) $P[X > 1/4]$

(e) $P[X \le 1/2]$

[Montgomery and Runger, 2010, Q3-42]

Remark: Try to calculate these values directly from the cdf. (Avoid converting the cdf to pmf first.)

Solution:

(a) $P[X \le 1/18] = F_X(1/18) = \boxed{0}$ because $\frac{1}{18} < \frac{1}{8}$.

(b) $P[X \le 1/4] = F_X(1/4) = \boxed{0.9}$.

(c) $P[X \le 5/16] = F_X(5/16) = \boxed{0.9}$ because $\frac{1}{4} < \frac{5}{16} < \frac{3}{8}$.

(d) $P[X > 1/4] = 1 - P[X \le 1/4] = 1 - F_X(1/4) = 1 - 0.9 = \boxed{0.1}$.

(e) $P[X \le 1/2] = F_X(1/2) = \boxed{1}$ because $\frac{1}{2} > \frac{3}{8}$.

Alternatively, we can also derive the pmf first and then calculate the probabilities.

Problem 3. [F2013/1] For each of the following random variables, find $P[1 < X \le 2]$.

(a) $X \sim \text{Binomial}(3, 1/3)$

(b) $X \sim \text{Poisson}(3)$

Solution:

(a) Because $X \sim \text{Binomial}(3, 1/3)$, we know that X can only take the values 0, 1, 2, 3. Only the value 2 satisfies the condition given. Therefore, $P[1 < X \le 2] = P[X = 2] = p_X(2)$. Recall that the pmf for the binomial random variable is

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, 3, \ldots, n$. Here, it is given that $n = 3$ and $p = 1/3$. Therefore,

$$p_X(2) = \binom{3}{2} \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right)^{3-2} = 3 \times \frac{1}{9} \times \frac{2}{3} = \boxed{\frac{2}{9}}.$$

(b) Because $X \sim \text{Poisson}(3)$, we know that X can take the values $0, 1, 2, 3, \ldots$. As in the previous part, only the value 2 satisfies the condition given. Therefore, $P[1 < X \le 2] = P[X = 2] = p_X(2)$. Recall that the pmf for the Poisson random variable is

$$p_X(x) = e^{-\alpha} \frac{\alpha^x}{x!}$$

for $x = 0, 1, 2, 3, \ldots$. Here, it is given that $\alpha = 3$. Therefore,

$$p_X(2) = e^{-3} \frac{3^2}{2!} = \boxed{\frac{9}{2} e^{-3} \approx 0.2240}.$$

Problem 4. Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 2$ customers per minute. Let M be the number of customers arriving between 9:00 and 9:05. What is the probability that $M < 2$?

Solution: Here, we are given that $M \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 2 \times 5 = 10$. Recall that, for $M \sim \mathcal{P}(\alpha)$, we have

$$P[M = m] = \begin{cases} e^{-\alpha}\frac{\alpha^m}{m!}, & m \in \{0, 1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P[M < 2] &= P[M = 0] + P[M = 1] = e^{-\alpha}\frac{\alpha^0}{0!} + e^{-\alpha}\frac{\alpha^1}{1!} \\ &= e^{-\alpha}(1 + \alpha) = e^{-10}(1 + 10) = 11e^{-10} \approx 5 \times 10^{-4}. \end{aligned}$$

Problem 5. [M2011/1] The cdf of a random variable X is plotted in Figure 8.3.

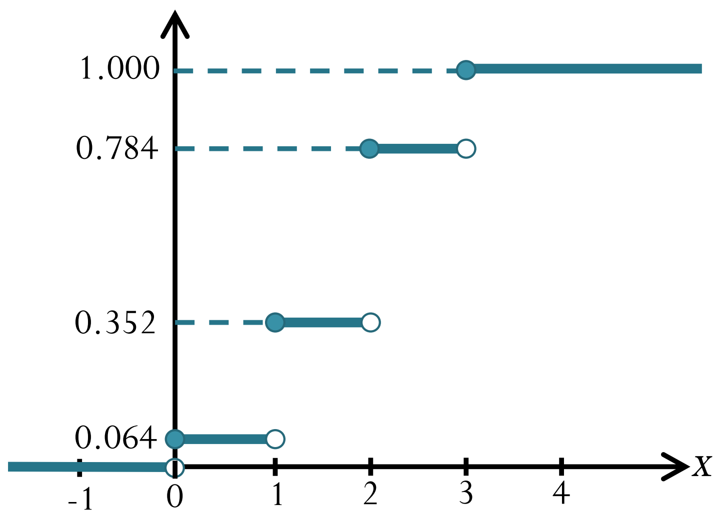

Figure 8.3: CDF of X for Problem 5

(a) Find the pmf $p_X(x)$.

(b) Find the family to which X belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

Solution:

(a) For discrete random variable, $P[X = x]$ is the jump size at x on the cdf plot. In this problem, there are four jumps at 0, 1, 2, 3.

- $P[X=0]$ = the jump size at $0 = 0.064 = \frac{64}{1000} = (4/10)^3 = (2/5)^3$.
- $P[X=1]$ = the jump size at $1 = 0.352 - 0.064 = 0.288$.
- $P[X=2]$ = the jump size at $2 = 0.784 - 0.352 = 0.432$.
- $P[X=3]$ = the jump size at $3 = 1 - 0.784 = 0.216 = (6/10)^3$.

In conclusion,

$$\boxed{p_X(x) = \begin{cases} 0.064, & x = 0, \\ 0.288, & x = 1, \\ 0.432, & x = 2, \\ 0.216, & x = 3, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) Among all the pmf that we discussed in class, only binomial pmf can have support $= \{0, 1, 2, 3\}$ with unequal probabilities. To check that the RV really is binomial, recall that the pmf for binomial X is given by $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ for $x = 0, 1, 2, \dots, n$. Here, $n = 3$. Furthermore, observe that $p_X(0) = (1-p)^n$. By comparing $p_X(0)$ with what we had in part (a), we have $1 - p = 2/5$ or $p = 3/5$. For $x = 1, 2, 3$, plugging in $p = 3/5$ and $n = 3$ in to $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ gives the same values as what we had in part (a). So, $\boxed{X \sim \mathcal{B}\left(3, \frac{3}{5}\right)}$.

Problem 6. When n is large, binomial distribution Binomial(n, p) becomes difficult to compute directly . In this question, we will consider an approximation when the value of p is close to 0. In such case, the binomial can be approximated[1] by the Poisson distribution with parameter $\alpha = np$. For this approximation to work, we will see in this exercise that n does not have to be very large and p does not need to be very small.

(a) Let $X \sim$ Binomial$(12, 1/36)$. (For example, roll two dice 12 times and let X be the number of times a double 6 appears.) Evaluate $p_X(x)$ for $x = 0, 1, 2$.

(b) Compare your answers part (a) with its Poisson approximation.

Solution:

[1] More specifically, suppose X_n has a binomial distribution with parameters n and p_n. If $p_n \to 0$ and $np_n \to \alpha$ as $n \to \infty$, then

$$P[X_n = k] \to e^{-\alpha} \frac{\alpha^k}{k!}.$$

(a) For Binomial(n, p) random variable,

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we are given that $n = 12$ and $p = \frac{1}{36}$. Plugging in $x = 0, 1, 2$, we get $\boxed{0.7132, 0.2445, 0.0384}$, respectively

(b) A Poisson random variable with parameter $\alpha = np$ can approximate a Binomial(n, p) random variable when n is large and p is small. Here, with $n = 12$ and $p = \frac{1}{36}$, we have $\alpha = 12 \times \frac{1}{36} = \frac{1}{3}$. The Poisson pmf at $x = 0, 1, 2$ is given by $e^{-\alpha} \frac{\alpha^x}{x!} = e^{-1/3} \frac{(1/3)^x}{x!}$. Plugging in $x = 0, 1, 2$ gives $\boxed{0.7165, 0.2388, 0.0398}$, respectively.

Figure 8.4 compares the two pmfs. Note how close they are!

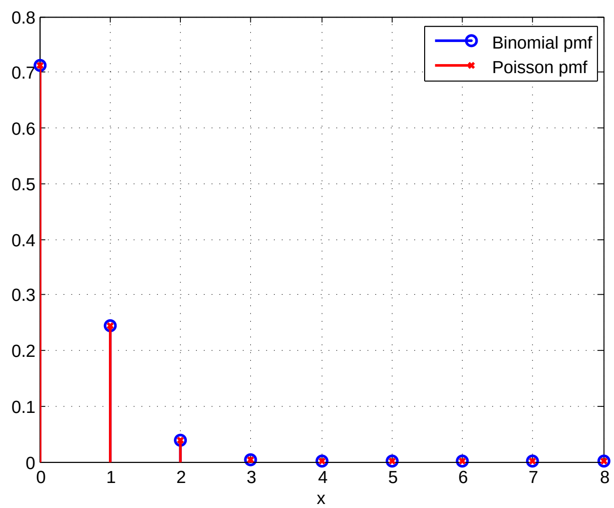

Figure 8.4: Poisson Approximation

Problem 7. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

Solution: Let N be the number of guests that has the same birthday as you. We may think of the comparison of your birthday with each of the guests as a Bernoulli trial. Here, there are 500 guests and therefore we are considering $n = 500$ trials. For each trial, the (success) probability that you have the same birthday as the corresponding guest is $p = \frac{1}{365}$. Then, this $N \sim$ Binomial(n, p).

(a) Binomial: $P[N=1] = np^1(1-p)^{n-1} \approx \boxed{0.348}$.

(b) Poisson: $P[N=1] = e^{-np}\frac{(np)^1}{1!} \approx \boxed{0.348}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 8. A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

(a) exactly one is emitted,

(b) more than three are emitted,

(c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

Solution: Let X be the number or particles emitted during the one second under consideration. Then $X \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 0.7 \times 1 = 0.7$.

(a) $P[X=1] = e^{-\alpha}\frac{\alpha^1}{1!} = \alpha e^{-\alpha} = 0.7e^{-0.7} \approx \boxed{0.3477}$.

(b) $P[X>3] = 1 - P[X \le 3] = 1 - \sum_{k=0}^{3} e^{-0.7}\frac{0.7^k}{k!} \approx \boxed{0.0058}$.

(c) $P[1 \le X \le 4] = \sum_{k=1}^{4} e^{-0.7}\frac{0.7^k}{k!} \approx \boxed{0.5026}$.

Problem 9 (M2011/1)**.** You are given an unfair coin with probability of obtaining a heads equal to $1/3,000,000,000$. You toss this coin 6,000,000,000 times. Let A be the event that you get "tails for all the tosses". Let B be the event that you get "heads for all the tosses".

(a) Approximate $P(A)$.

(b) Approximate $P(A \cup B)$.

Solution: Let N be the number of heads among the n tosses. Then, $N \sim \mathcal{B}(n,p)$. Here, we have small $p = 1/3 \times 10^9$ and large $n = 6 \times 10^9$. So, we can apply Poisson approximation. In other words, $\mathcal{B}(n,p)$ is well-approximated by $\mathcal{P}(\alpha)$ where $\alpha = np = 2$.

(a) $P(A) = P[N=0] = e^{-2}\frac{2^0}{0!} = \frac{1}{e^2} \approx \boxed{0.1353}$.

(b) Note that events A and B are disjoint. Therefore, $P(A \cup B) = P(A) + P(B)$. We have already calculated $P(A)$ in the previous part. For $P(B)$, from $N \sim \mathcal{B}(n,p)$, we have $P(B) = P[N=n] = p^n = \left(\frac{1}{3\times 10^9}\right)^{6\times 10^9}$. Observe that $P(B)$ is extremely small compared to $P(A)$. Therefore, $P(A \cup B)$ is approximately the same as $P(A) \approx \boxed{0.1353}$.

EES 315: Probability and Random Processes **2020/1**

HW Solution 9 — Due: November 25, 11:59 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Consider a random variable X whose pmf is

$$p_X(x) = \begin{cases} 1/2, & x = -1, \\ 1/4, & x = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2$.

(a) Find $\mathbb{E}X$.

(b) Find $\mathbb{E}[X^2]$.

(c) Find $\operatorname{Var} X$.

(d) Find σ_X.

(e) Find $p_Y(y)$.

(f) Find $\mathbb{E}Y$.

(g) Find $\mathbb{E}[Y^2]$.

Solution:

(a) $\mathbb{E}X = \sum_x x p_X(x) = (-1) \times \frac{1}{2} + (0) \times \frac{1}{4} + (1) \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{4} = \boxed{-\frac{1}{4}}$.

(b) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = (-1)^2 \times \frac{1}{2} + (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$.

(c) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{3}{4} - \left(-\frac{1}{4}\right)^2 = \frac{3}{4} - \frac{1}{16} = \boxed{\frac{11}{16}}$.

(d) $\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\frac{\sqrt{11}}{4}}$.

(e) First, we build a table to see which values y of Y are possible from the values x of X:

x	$p_X(x)$	y
-1	1/2	$(-1)^2 = 1$
0	1/4	$(0)^2 = 0$
1	1/4	$(1)^2 = 1$

Therefore, the random variable Y can takes two values: 0 and 1. $p_Y(0) = p_X(0) = 1/4$. $p_Y(1) = p_X(-1) + p_X(1) = 1/2 + 1/4 = 3/4$. Therefore,

$$\boxed{p_Y(y) = \begin{cases} 1/4, & y = 0, \\ 3/4, & y = 1, \\ 0, & \text{otherwise.} \end{cases}}$$

(f) $\mathbb{E}Y = \sum_y y p_Y(y) = (0) \times \frac{1}{4} + (1) \times \frac{3}{4} = \boxed{\frac{3}{4}}$. Alternatively, because $Y = X^2$, we automatically have $\mathbb{E}[Y] = \mathbb{E}[X^2]$. Therefore, we can simply use the answer from part (b).

(g) $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{3}{4} = \boxed{\frac{3}{4}}$. Alternatively,

$$\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[X^4\right] = \sum_x x^4 p_X(x) = (-1)^4 \times \frac{1}{2} + (0)^4 \times \frac{1}{4} + (1)^4 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Problem 2. For each of the following random variables, find $\mathbb{E}X$ and σ_X.

(a) $X \sim$ Binomial$(3, 1/3)$

(b) $X \sim$ Poisson(3)

Solution:

(a) From the lecture notes, we know that when $X \sim$ Binomial(n, p), we have $\mathbb{E}X = np$ and $\operatorname{Var} X = np(1-p)$. Here, $n = 3$ and $p = 1/3$. Therefore, $\mathbb{E}X = 3 \times \frac{1}{3} = \boxed{1}$. Also, because $\operatorname{Var} X = 3\left(\frac{1}{3}\right)\left(1 - \frac{1}{3}\right) = \frac{2}{3}$, we have $\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\sqrt{\frac{2}{3}}}$.

(b) From the lecture notes, we know that when $X \sim$ Poisson(α), we have $\mathbb{E}X = \alpha$ and $\operatorname{Var} X = \alpha$. Here, $\alpha = 3$. Therefore, $\mathbb{E}X = \boxed{3}$. Also, because $\operatorname{Var} X = 3$, we have $\sigma_X = \boxed{\sqrt{3}}$.

Problem 3. Suppose X is a uniform discrete random variable on $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Find

(a) $\mathbb{E}X$

(b) $\mathbb{E}\left[X^2\right]$

(c) $\operatorname{Var} X$

(d) σ_X

Solution: All of the calculations in this question are simply plugging in numbers into appropriate formulas.

(a) $\mathbb{E}X = \boxed{0.5}$

(b) $\mathbb{E}\left[X^2\right] = \boxed{5.5}$

(c) $\operatorname{Var} X = \boxed{5.25}$

(d) $\sigma_X = \boxed{2.2913}$

Alternatively, we can find a formula for the general case of uniform random variable X on the sets of integers from a to b. Note that there are $n = b - a + 1$ values that the random variable can take. Hence, all of them has probability $\frac{1}{n}$.

(a) $\mathbb{E}X = \sum_{k=a}^{b} k\frac{1}{n} = \frac{1}{n}\sum_{k=a}^{b} k = \frac{1}{n} \times \frac{n(a+b)}{2} = \frac{a+b}{2}$.

(b) First, note that

$$\begin{aligned}\sum_{i=a}^{b} k\left(k-1\right) &= \sum_{k=a}^{b} k\left(k-1\right)\left(\frac{(k+1)-(k-2)}{3}\right)\\ &= \frac{1}{3}\left(\sum_{k=a}^{b}(k+1)\,k\,(k-1) - \sum_{k=a}^{b} k\,(k-1)\,(k-2)\right)\\ &= \frac{1}{3}\left((b+1)\,b\,(b-1) - a\,(a-1)\,(a-2)\right)\end{aligned}$$

where the last equality comes from the fact that there are many terms in the first sum that is repeated in the second sum and hence many cancellations.

Now,

$$\begin{aligned}\sum_{k=a}^{b} k^2 = \sum_{k=a}^{b}\left(k\left(k-1\right)+k\right) &= \sum_{k=a}^{b} k\left(k-1\right) + \sum_{k=a}^{b} k\\ &= \frac{1}{3}\left((b+1)\,b\,(b-1) - a\,(a-1)\,(a-2)\right) + \frac{n\,(a+b)}{2}\end{aligned}$$

Therefore,

$$\sum_{k=a}^{b} k^2 \frac{1}{n} = \frac{1}{3n}\left((b+1)\,b\,(b-1) - a\,(a-1)\,(a-2)\right) + \frac{a+b}{2}$$
$$= \frac{1}{3}\,a^2 - \frac{1}{6}\,a + \frac{1}{3}\,ab + \frac{1}{6}\,b + \frac{1}{3}\,b^2$$

(c) $\operatorname{Var} X = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = \frac{1}{12}\,(b-a)\,(b-a+2) = \frac{1}{12}(n-1)(n+1) = \frac{n^2-1}{12}$.

(d) $\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{\frac{n^2-1}{12}}$.

Problem 4. (Expectation + pmf + Gambling + Effect of miscalculation of probability) In the eighteenth century, a famous French mathematician Jean Le Rond d'Alembert, author of several works on probability, analyzed the toss of two coins. He reasoned that because this experiment has THREE outcomes, (the number of heads that turns up in those two tosses can be 0, 1, or 2), the chances of each must be 1 in 3. In other words, if we let N be the number of heads that shows up, Alembert would say that

$$p_N(n) = 1/3 \quad \text{for} \quad N = 0, 1, 2. \tag{9.1}$$

[Mlodinow, 2008, p 50–51]

We know that Alembert's conclusion was ***wrong***. His three outcomes are not equally likely and hence classical probability formula can not be applied directly. The key is to realize that there are FOUR outcomes which are equally likely. We should not consider 0, 1, or 2 heads as the possible outcomes. There are in fact four equally likely outcomes: (heads, heads), (heads, tails), (tails, heads), and (tails, tails). These are the 4 possibilities that make up the sample space. The actual pmf for N is

$$p_N(n) = \begin{cases} 1/4, & n = 0, 2, \\ 1/2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose you travel back in time and meet Alembert. You could make the following bet with Alembert to gain some easy money. The bet is that if the result of a toss of two coins contains exactly one head, then he would pay you \$150. Otherwise, you would pay him \$100.

Let R be Alembert's profit from this bet and Y be the your profit from this bet.

(a) Then, $R = -150$ if you win and $R = +100$ otherwise. Use Alembert's *miscalculated* probabilities from (9.1) to determine the pmf of R (from Alembert's belief).

(b) Use Alembert's *miscalculated* probabilities from (9.1) (or the corresponding (miscalculated) pmf found in part (a)) to calculate $\mathbb{E}R$, the expected profit for Alembert.

Remark: You should find that $\mathbb{E}R > 0$ and hence Alembert will be quite happy to accept your bet.

(c) Use the *actual* probabilities, to determine the pmf of R.

(d) Use the *actual* pmf, to determine $\mathbb{E}R$.

Remark: You should find that $\mathbb{E}R < 0$ and hence Alembert should not accept your bet if he calculates the probabilities correctly.

(e) Note that $Y = +150$ if you win and $Y = -100$ otherwise. Use the *actual* probabilities to determine the pmf of Y.

(f) Use the *actual* probabilities, to determine $\mathbb{E}Y$.

Remark: You should find that $\mathbb{E}Y > 0$. This is the amount of money that you expect to gain each time that you play with Alembert. Of course, Alembert, who still believes that his calculation is correct, will ask you to play this bet again and again believing that he will make profit in the long run.

By miscalculating probabilities, one can make wrong decisions (and lose a lot of money)!
Solution:

(a) $P[R = -150] = P[N = 1]$ and $P[R = +100] = P[N \neq 1] = P[N = 0] + P[N = 2]$. So,

$$\boxed{p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise.} \end{cases}}$$

Using Alembert's *miscalculated* pmf,

$$p_R(r) = \begin{cases} 1/3, & r = -150, \\ 2/3, & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

(b) From $p_R(r)$ in part (a), we have $\mathbb{E}R = \sum_r p_R(r) = \frac{1}{3} \times (-150) + \frac{2}{3} \times 100 = \boxed{\frac{50}{3}} \approx 16.67$

(c) Again,

$$p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

Using the actual pmf,

$$p_R(r) = \begin{cases} \frac{1}{2}, & r = -150, \\ \frac{1}{4} + \frac{1}{4}, & r = +100, \\ 0, & \text{otherwise} \end{cases} = \boxed{\begin{cases} \frac{1}{2}, & r = -150 \text{ or } +100, \\ 0, & \text{otherwise.} \end{cases}}$$

(d) From $p_R(r)$ in part (c), we have $\mathbb{E}R = \sum_r p_R(r) = \frac{1}{2} \times (-150) + \frac{1}{2} \times 100 = \boxed{-25}$.

(e) Observe that $Y = -R$. Hence, using the answer from part (c), we have

$$p_Y(y) = \boxed{\begin{cases} \frac{1}{2}, & y = +150 \text{ or } -100, \\ 0, & \text{otherwise.} \end{cases}}$$

(f) Observe that $Y = -R$. Hence, $\mathbb{E}Y = -\mathbb{E}R$. Using the actual probabilities, $\mathbb{E}R = -25$ from part (d). Hence, $\mathbb{E}Y = \boxed{+25}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. A random variables X has support containing only two numbers. Its expected value is $\mathbb{E}X = 5$. Its variance is $\operatorname{Var} X = 3$. Give an example of the pmf of such a random variable.

Solution: We first find $\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{3}$. Recall that this is the average deviation from the mean. Because X takes only two values, we can make them at exactly $\pm\sqrt{3}$ from the mean; that is

$$x_1 = 5 - \sqrt{3} \quad \text{and} \quad x_2 = 5 + \sqrt{3}.$$

In which case, we automatically have $\mathbb{E}X = 5$ and $\operatorname{Var} X = 3$. Hence, one example of such pmf is

$$\boxed{p_X(x) = \begin{cases} \frac{1}{2}, & x = 5 \pm \sqrt{3} \\ 0, & \text{otherwise} \end{cases}}$$

We can also try to find a general formula for x_1 and x_2. If we let $p = P[X = x_2]$, then $q = 1 - p = P[X = x_1]$. Given p, the values of x_1 and x_2 must satisfy two conditions: $\mathbb{E}X = m$ and $\operatorname{Var} X = \sigma^2$. (In our case, $m = 5$ and $\sigma^2 = 3$.) From $\mathbb{E}X = m$, we must have

$$x_1 q + x_2 p = m; \tag{9.2}$$

that is

$$x_1 = \frac{m}{q} - x_2\frac{p}{q}.$$

From $\operatorname{Var} X = \sigma^2$, we have $\mathbb{E}\left[X^2\right] = \operatorname{Var} X + \mathbb{E}X^2 = \sigma^2 + m^2$ and hence we must have

$$x_1^2 q + x_2^2 p = \sigma^2 + m^2. \tag{9.3}$$

Substituting x_1 from (9.2) into (9.3), we have

$$x_2^2 p - 2x_2 mp + \left(pm^2 - q\sigma^2\right) = 0$$

whose solutions are

$$x_2 = \frac{2mp \pm \sqrt{4m^2p^2 - 4p\left(pm^2 - q\sigma^2\right)}}{2p} = \frac{2mp \pm 2\sigma\sqrt{pq}}{2p} = m \pm \sigma\sqrt{\frac{q}{p}}.$$

Using (9.2), we have

$$x_1 = \frac{m}{q} - \left(m \pm \sigma\sqrt{\frac{q}{p}}\right)\frac{p}{q} = m \mp \sigma\sqrt{\frac{p}{q}}.$$

Therefore, for any given p, there are two pmfs:

$$p_X(x) = \begin{cases} 1-p, & x = m - \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m + \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise,} \end{cases}$$

or

$$p_X(x) = \begin{cases} 1-p, & x = m + \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m - \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 6. For each of the following families of random variable X, find the value(s) of x which maximize $p_X(x)$. (This can be interpreted as the "mode" of X.)

(a) $\mathcal{P}(\alpha)$

(b) Binomial(n,p)

(c) $\mathcal{G}_0(\beta)$

(d) $\mathcal{G}_1(\beta)$

Remark [Y&G, p. 66]:

- For statisticians, the mode is the most common number in the collection of observations. There are as many or more numbers with that value than any other value. If there are two or more numbers with this property, the collection of observations is called multimodal. In probability theory, a **mode** of random variable X is a number x_{mode} satisfying
$$p_X(x_{\text{mode}}) \geq p_X(x) \quad \text{for all } x.$$
- For statisticians, the median is a number in the middle of the set of numbers, in the sense that an equal number of members of the set are below the median and above the median. In probability theory, a median, X_{median}, of random variable X is a number that satisfies
$$P[X < X_{\text{median}}] = P[X > X_{\text{median}}].$$
- Neither the mode nor the median of a random variable X need be unique. A random variable can have several modes or medians.

Solution: We first note that when $\alpha > 0$, $p \in (0,1)$, $n \in \mathbb{N}$, and $\beta \in (0,1)$, the above pmf's will be strictly positive for some values of x. Hence, we can discard those x at which $p_X(x) = 0$. The remaining points are all integers. To compare them, we will evaluate $\frac{p_X(i+1)}{p_X(i)}$.

(a) For Poisson pmf, we have
$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{e^{-\alpha}\alpha^{i+1}}{(i+1)!}}{\frac{e^{-\alpha}\alpha^{i}}{i!}} = \frac{\alpha}{i+1}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$ if and only if $i < \alpha - 1$.
- $\frac{p_X(i+1)}{p_X(i)} = 1$ if and only if $i = \alpha - 1$.
- $\frac{p_X(i+1)}{p_X(i)} < 1$ if and only if $i > \alpha - 1$.

Let $\tau = \alpha - 1$. This implies that τ is the place where things change. Moving from i to $i+1$, the probability strictly increases if $i < \tau$. When $i > \tau$, the next probability value (at $i+1$) will decrease.

(i) Suppose $\alpha \in (0,1)$, then $\alpha - 1 < 0$ and hence $i > \alpha - 1$ for all i. (Note that i are are nonnegative integers.) This implies that the pmf is a strictly decreasing function and hence the maximum occurs at the first i which is $i = 0$.

(ii) Suppose $\alpha \in \mathbb{N}$. Then, the pmf will be strictly increasing until we reaches $i = \alpha - 1$. At which point, the next probability value is the same. Then, as we further increase i, the pmf is strictly decreasing. Therefore, the maximum occurs at $\alpha - 1$ and α.

(iii) Suppose $\alpha \notin \mathbb{N}$ and $\alpha \geq 1$. Then we will have have any $i = \alpha - 1$. The pmf will be strictly increasing where the last increase is from $i = \lfloor \alpha - 1 \rfloor$ to $i + 1 = \lfloor \alpha - 1 \rfloor + 1 = \lfloor \alpha \rfloor$. After this, the pmf is strictly decreasing. Hence, the maximum occurs at $\lfloor \alpha \rfloor$.

To summarize,

$$\arg\max_x p_X(x) = \boxed{\begin{cases} 0, & \alpha \in (0,1), \\ \alpha - 1 \text{ and } \alpha, & \alpha \text{ is an integer}, \\ \lfloor \alpha \rfloor, & \alpha > 1 \text{ is not an integer}. \end{cases}}$$

(b) For binomial pmf, we have

$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{n!}{(i+1)!(n-i-1)!} p^{i+1} (1-p)^{n-i-1}}{\frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}} = \frac{(n-i)\,p}{(i+1)(1-p)}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$ if and only if $i < np - 1 + p = (n+1)p - 1$.
- $\frac{p_X(i+1)}{p_X(i)} = 1$ if and only if $i = (n+1)p - 1$.
- $\frac{p_X(i+1)}{p_X(i)} < 1$ if and only if $i > (n+1)p - 1$.

Let $\tau = (n+1)p - 1$. This implies that τ is the place where things change. Moving from i to $i+1$, the probability strictly increases if $i < \tau$. When $i > \tau$, the next probability value (at $i+1$) will decrease.

(i) Suppose $(n+1)p$ is an integer. The pmf will strictly increase as a function of i, and then stays at the same value at $i = \tau = (n+1)p - 1$ and $i + 1 = (n+1)p - 1 + 1 = (n+1)p$. Then, it will strictly decrease. So, the maximum occurs at $(n+1)p - 1$ and $(n+1)p$.

(ii) Suppose $(n+1)p$ is not an integer. Then, there will not be any i that is $= \tau$. Therefore, we only have the pmf strictly increases where the last increase occurs when we goes from $i = \lfloor \tau \rfloor$ to $i + 1 = \lfloor \tau \rfloor + 1$. After this, the probability is strictly decreasing. Hence, the maximum is unique and occur at $\lfloor \tau \rfloor + 1 = \lfloor (n+1)p - 1 \rfloor + 1 = \lfloor (n+1)p \rfloor$.

To summarize,

$$\arg\max_x p_X(x) = \boxed{\begin{cases} (n+1)p - 1 \text{ and } (n+1)p, & (n+1)p \text{ is an integer}, \\ \lfloor (n+1)p \rfloor, & (n+1)p \text{ is not an integer}. \end{cases}}$$

(c) $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$. Hence, $p_X(i)$ is strictly decreasing. The maximum occurs at the smallest value of i which is $\boxed{0}$.

(d) $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$. Hence, $p_X(i)$ is strictly decreasing. The maximum occurs at the smallest value of i which is $\boxed{1}$.

Problem 7. An article in Information Security Technical Report ["Malicious Software—Past, Present and Future" (2004, Vol. 9, pp. 618)] provided the data (shown in Figure 9.1) on the top ten malicious software instances for 2002. The clear leader in the number of registered incidences for the year 2002 was the Internet worm "Klez". This virus was first detected on 26 October 2001, and it has held the top spot among malicious software for the longest period in the history of virology.

Place	Name	% Instances
1	I-Worm.Klez	61.22%
2	I-Worm.Lentin	20.52%
3	I-Worm.Tanatos	2.09%
4	I-Worm.BadtransII	1.31%
5	Macro.Word97.Thus	1.19%
6	I-Worm.Hybris	0.60%
7	I-Worm.Bridex	0.32%
8	I-Worm.Magistr	0.30%
9	Win95.CIH	0.27%
10	I-Worm.Sircam	0.24%

Figure 9.1: The 10 most widespread malicious programs for 2002 (Source—Kaspersky Labs).

Suppose that 20 malicious software instances are reported. Assume that the malicious sources can be assumed to be independent.

(a) What is the probability that at least one instance is "Klez"?

(b) What is the probability that three or more instances are "Klez"?

(c) What are the expected value and standard deviation of the number of "Klez" instances among the 20 reported?

Solution: Let N be the number of instances (among the 20) that are "Klez". Then, $N \sim$binomial(n, p) where $n = 20$ and $p = 0.6122$.

(a) $P[N \geq 1] = 1 - P[N < 1] = 1 - P[N = 0] = 1 - p_N(0) = 1 - \binom{20}{0} \times 0.6122^0 \times 0.3878^{20} \approx 0.9999999941 \approx 1$.

(b)
$$\begin{aligned} P[N \geq 3] &= 1 - P[N < 3] = 1 - (P[N = 0] + P[N = 1] + P[N = 2]) \\ &= 1 - \sum_{k=0}^{2} \binom{20}{k} (0.6122)^k (0.3878)^{20-k} \approx 0.999997 \end{aligned}$$

(c) $\mathbb{E}N = np = 20 \times 0.6122 = 12.244$.
$\sigma_N = \sqrt{\operatorname{Var} N} = \sqrt{np(1-p)} = \sqrt{20 \times 0.6122 \times 0.3878} \approx 2.179$.

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HW Solution 10 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1 (Yates and Goodman, 2005, Q3.2.1)**.** The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Use the pdf to find the following quantities.

(a) the unknown constant c

(b) $P[0 \le X \le 1]$

(c) $P[-1/2 \le X \le 1/2]$.

Solution:

(a) Recall that any pdf should integrate to 1. Here,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 cx\,dx = c\left.\frac{x^2}{2}\right|_0^2 = 2c.$$

Equating the expression above to 1, we get $c = \boxed{\frac{1}{2}}$.

(b) $P[0 \le X \le 1] = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^1 = \boxed{\frac{1}{4}}$.

(c) $P\left[-\frac{1}{2} \le X \le \frac{1}{2}\right] = \int_{-1/2}^{1/2} f_X(x)\,dx$. Now, $f_X(x) = 0$ on the interval $\left[-\frac{1}{2}, 0\right)$. Therefore, we don't have to integrate over that interval and

$$P\left[-\frac{1}{2} \le X \le \frac{1}{2}\right] = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^{1/2} = \boxed{\frac{1}{16}}.$$

Problem 2 (Yates and Goodman, 2005, Q3.3.4)**.** The pdf of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[Y]$.

(b) Find $\operatorname{Var} Y$.

Solution:

(a) Recall that, for continuous random variable Y,

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Note that when y is outside of the interval $[0, 2)$, $f_Y(y) = 0$ and hence does not affect the integration. We only need to integrate over $[0, 2)$ in which $f_Y(y) = \frac{y}{2}$. Therefore,

$$\mathbb{E}Y = \int_0^2 y\left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^2}{2} dy = \left.\frac{y^3}{2 \times 3}\right|_0^2 = \boxed{\frac{4}{3}}.$$

(b) The variance of any random variable Y (discrete or continuous) can be found from

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2$$

We have already calculate $\mathbb{E}Y$ in the previous part. So, now we need to calculate $\mathbb{E}[Y^2]$. Recall that, for continuous random variable,

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y) dy.$$

Here, $g(y) = y^2$. Therefore,

$$\mathbb{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy.$$

Again, in the integration, we can ignore the y whose $f_Y(y) = 0$:

$$\mathbb{E}\left[Y^2\right] = \int_0^2 y^2\left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^3}{2} dy = \left.\frac{y^4}{2 \times 4}\right|_0^2 = \boxed{2}.$$

Plugging this into the variance formula gives

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}.$$

Problem 3. The pdf of random variable V is

$$f_V(v) = \begin{cases} \frac{v+5}{72}, & -5 < v < 7, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is $\mathbb{E}[V]$?

(b) What is $\text{Var}[V]$?

(c) What is $\mathbb{E}[V^3]$?

Solution:

(a) $\mathbb{E}[V] = \int_{-\infty}^{\infty} v f_V(v)\, dv = \int_{-5}^{7} v\left(\frac{v+5}{72}\right) dv = \frac{1}{72}\int_{-5}^{7} v^2 + 5v dv = \boxed{3}$.

(b) $\mathbb{E}[V^2] = \int_{-\infty}^{\infty} v^2 f_V(v)\, dv = \int_{-5}^{7} v^2\left(\frac{v+5}{72}\right) dv = 17.$
Therefore, $\text{Var}\, V = \mathbb{E}[V^2] - (\mathbb{E}[V])^2 = 17 - 9 = \boxed{8}$.

(c) $\mathbb{E}[V^3] = \int_{-\infty}^{\infty} v^3 f_V(v)\, dv = \int_{-5}^{7} v^3\left(\frac{v+5}{72}\right) dv = \boxed{\frac{431}{5} = 86.2}$.

Problem 4 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

(b) What is $\mathbb{E}[X]$?

(c) What is $\mathbb{E}[X^5]$?

(d) What is $\mathbb{E}\left[e^X\right]$?

Solution: For a uniform random variable X on the interval (a, b), we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

In this problem, we have $a = -5$ and $b = 5$.

(a) $f_X(x) = \boxed{\begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \le x \le 5 \end{cases}}$

(b) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\, dx = \int\limits_{-5}^{5} x \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^2}{2}\right|_{-5}^{5} = \frac{1}{20}\left(5^2 - (-5)^2\right) = \boxed{0}.$

In general,

$$\mathbb{E}X = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a} \frac{b^2-a^2}{2} = \frac{a+b}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}X = \boxed{0}$.

(c) $\mathbb{E}\left[X^5\right] = \int\limits_{-\infty}^{\infty} x^5 f_X(x)\, dx = \int\limits_{-5}^{5} x^5 \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^6}{6}\right|_{-5}^{5} = \frac{1}{60}\left(5^6 - (-5)^6\right) = \boxed{0}.$

In general,

$$\mathbb{E}\left[X^5\right] = \int_a^b x^5 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^5 dx = \frac{1}{b-a} \left.\frac{x^6}{6}\right|_a^b = \frac{1}{b-a} \frac{b^6-a^6}{6}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[X^5\right] = \boxed{0}$.

(d) In general,

$$\mathbb{E}\left[e^X\right] = \int_a^b e^x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b e^x dx = \frac{1}{b-a} \left. e^x \right|_a^b = \frac{e^b - e^a}{b-a}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[e^X\right] = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84.$

Problem 5 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution: First, because Θ is a uniform random variable on the interval $(0,2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi}1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}\left[g(\Theta)\right] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t+\Theta)] = 5\int_0^{2\pi}\frac{1}{2\pi}\cos(7t+\theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

(b) Y is another function of Θ.

$$\mathbb{E}[Y] = \mathbb{E}\left[5\cos(7t_1+\Theta)\times 5\cos(7t_2+\Theta)\right] = \int_0^{2\pi}\frac{1}{2\pi}5\cos(7t_1+\theta)\times 5\cos(7t_2+\theta)d\theta$$
$$= \frac{25}{2\pi}\int_0^{2\pi}\cos(7t_1+\theta)\times\cos(7t_2+\theta)d\theta.$$

Recall[1] the cosine identity

$$\cos(a)\times\cos(b) = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right).$$

Therefore,

$$\mathbb{E}Y = \frac{25}{4\pi}\int_0^{2\pi}\cos(7t_1+7t_2+2\theta)+\cos(7(t_1-t_2))\,d\theta$$
$$= \frac{25}{4\pi}\left(\int_0^{2\pi}\cos(7t_1+7t_2+2\theta)\,d\theta + \int_0^{2\pi}\cos(7(t_1-t_2))\,d\theta\right).$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1-t_2))\int_0^{2\pi}d\theta = \frac{25}{4\pi}\cos(7(t_1-t_2))\,2\pi = \boxed{\frac{25}{2}\cos(7(t_1-t_2))}.$$

[1]This identity could be derived easily via the Euler's identity:

$$\cos(a)\times\cos(b) = \frac{e^{ja}+e^{-ja}}{2}\times\frac{e^{jb}+e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb}+e^{-ja}e^{jb}+e^{ja}e^{-jb}+e^{-ja}e^{-jb}\right)$$
$$= \frac{1}{2}\left(\frac{e^{ja}e^{jb}+e^{-ja}e^{-jb}}{2}+\frac{e^{-ja}e^{jb}+e^{ja}e^{-jb}}{2}\right)$$
$$= \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right).$$

Problem 6. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

Solution: Let T be the time to failure (in hours). We are given that $T \sim \mathcal{E}(\lambda)$ where $\lambda = 3 \times 10^{-4}$. Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find $P[T > 10^4]$.

We shall first provide the general formula for the ccdf $P[T > t]$ when $t > 0$:

$$P[T > t] = \int_t^\infty f_T(\tau) d\tau = \int_t^\infty \lambda e^{-\lambda\tau} d\tau = -e^{-\lambda\tau}\Big|_t^\infty = e^{-\lambda t}. \tag{10.1}$$

Therefore,

$$P\left[T > 10^4\right] = e^{-3\times10^{-4}\times10^4} = \boxed{e^{-3} \approx 0.0498.}$$

(b) We start with $P[T \le 7000] = 1 - P[T > 7000]$. Next, we apply (10.1) to get

$$P[T \le 7000] = 1 - P[T > 7000] = 1 - e^{-3\times10^{-4}\times7000} = \boxed{1 - e^{-2.1} \approx 0.8775.}$$

Problem 7. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find the expected value of X.

(c) Find the variance and the standard deviation of X.

(d) Find the expected value of power when the resistance is 100 ohms?

Solution:

(a) $P[X < 5] = \int\limits_{-\infty}^{5} f_X(x)dx = \int\limits_{-\infty}^{0} \underbrace{f_X(x)}_{0}dx + \int\limits_{0}^{5} \underbrace{f_X(x)}_{5}dx = 0 + 5x\big|_{x=4.9}^{5} = \boxed{0.5}.$

(b) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\,dx = \int\limits_{-\infty}^{4.9} x \underbrace{f_X(x)}_{0} dx + \int\limits_{4.9}^{5.1} x \underbrace{f_X(x)}_{5} dx + \int\limits_{5.1}^{x} x \underbrace{f_X(x)}_{0} dx = 0 + 5\frac{x^2}{2}\Big|_{x=4.9}^{5.1} + 0 = \boxed{5}$ mA.

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\mathbb{E}X = \frac{a+b}{2} = \frac{4.9+5.1}{2} = 5$.

(c) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$. From the previous part, we know that $\mathbb{E}X = 5$. SO, to find $\operatorname{Var} X$, we need to find $\mathbb{E}[X^2]$: $\mathbb{E}[X^2] = \int\limits_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int\limits_{-\infty}^{4.9} x^2 \underbrace{f_X(x)}_{0} dx + \int\limits_{4.9}^{5.1} x^2 \underbrace{f_X(x)}_{5} dx + \int\limits_{5.1}^{x} x^2 \underbrace{f_X(x)}_{0} dx = 0 + 5\frac{x^3}{3}\Big|_{x=4.9}^{5.1} + 0 = 25 + \frac{1}{300}$.

Therefore, $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \left(25 + \frac{1}{300}\right) = \boxed{\frac{1}{300}} \approx 0.0033$ (mA)^2

and

$\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\frac{1}{10\sqrt{3}}} \approx 0.0577$ mA.

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\operatorname{Var} X = \frac{(b-a)^2}{12} = \frac{(5.1-4.9)^2}{12} = \frac{1}{300}$.

(d) Recall that $P = I \times V = I^2 r$. Here, $I = X$. Therefore, $P = X^2 r$ and $\mathbb{E}P = \mathbb{E}[X^2 r] = r\mathbb{E}[X^2] = 100 \times \left(25 + \frac{1}{300}\right) = 2500 + \frac{1}{3} \approx 2.50033 \times 10^3$ [(mA)$^2\Omega$]. Factoring out m^2, we have $\mathbb{E}P \approx 2.50033$ mW. ([A$^2\Omega$] = [W].)

Problem 8. Let X be a uniform random variable on the interval $[0,1]$. Set

$$A = \left[0, \frac{1}{2}\right), \quad B = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), \quad \text{and } C = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \cup \left[\frac{3}{4}, \frac{7}{8}\right).$$

Are the events $[X \in A], [X \in B]$, and $[X \in C]$ independent?

Solution: Note that

$$P[X \in A] = \int\limits_0^{\frac{1}{2}} dx = \frac{1}{2},$$

$$P[X \in B] = \int\limits_0^{\frac{1}{4}} dx + \int\limits_{\frac{1}{2}}^{\frac{3}{4}} dx = \frac{1}{2}, \text{ and}$$

$$P[X \in C] = \int\limits_0^{\frac{1}{8}} dx + \int\limits_{\frac{1}{4}}^{\frac{3}{8}} dx + \int\limits_{\frac{1}{2}}^{\frac{5}{8}} dx + \int\limits_{\frac{3}{4}}^{\frac{7}{8}} dx = \frac{1}{2}.$$

Now, for pairs of events, we have

$$P([X \in A] \cap [X \in B]) = \int_{0}^{\frac{1}{4}} dx = \frac{1}{4} = P[X \in A] \times P[X \in B], \tag{10.2}$$

$$P([X \in A] \cap [X \in C]) = \int_{0}^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx = \frac{1}{4} = P[X \in A] \times P[X \in C], \text{ and} \tag{10.3}$$

$$P([X \in B] \cap [X \in C]) = \int_{0}^{\frac{1}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx = \frac{1}{4} = P[X \in B] \times P[X \in C]. \tag{10.4}$$

Finally,

$$P([X \in A] \cap [X \in B] \cap [X \in C]) = \int_{0}^{\frac{1}{8}} dx = \frac{1}{8} = P[X \in A] P[X \in B] P[X \in C]. \tag{10.5}$$

From (10.2), (10.3), (10.4) and (10.5), we can conclude that the events $[X \in A], [X \in B]$, and $[X \in C]$ are $\boxed{\text{independent}}$.