

EES 315: In-Class Exercise # 11

Instructions

1. Work alone or in a group of no more than three students.
2. Only one submission is needed for each group.
3. You have two choices for submission:
 - (a) Online submission via Google Classroom
 - PDF only.
 - Only for those who can directly work on the posted files using devices with pen input.
 - Paper size should be the same as the posted file.
 - No scanned work, photos, or screen capture.
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(You may add the IDs of other members, exercise #, or other information as well.)
 - (b) Hardcopy submission
4. **Do not panic.**

Date: 7 / 10 / 2020			
Name			ID <small>(last 3 digits)</small>

(1) Suppose $P(A) = 0.5$ and $P(B) = 0.2$.

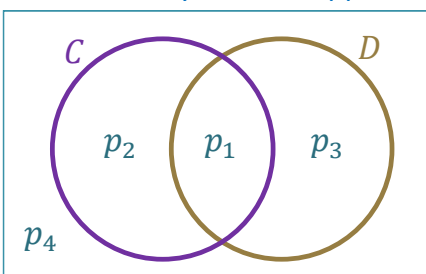
Find $P(A \cap B)$ to make events A and B independent.

By definition, events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.
Therefore, $P(A \cap B) = 0.5 \times 0.2 = 0.1$.

(2) Suppose $P(D) = 0.6$ and $P(C \cup D) = 0.8$.

Find $P(C \cap D)$ to make events C and D independent.

We will use a systematic approach. Consider the Venn diagram below.



As usual, for two events we partition the sample space (Ω) into 4 parts.

Let p_i be the probability of the i^{th} part.

Observe that $P(C \cap D)$ is p_1 .

From the provided information, we know that

$$\left. \begin{aligned} p_1 + p_3 &= P(D) = 0.6. & (1) \\ p_1 + p_2 + p_3 &= P(C \cup D) = 0.8. & (2) \end{aligned} \right\}$$

As usual, we also know that

$$p_1 + p_2 + p_3 + p_4 = 1. \quad (3)$$

The requirement that events C and D must be independent means

$$P(C \cap D) = P(C)P(D)$$

This is equivalent to

$$p_1 = (p_1 + p_2)(p_1 + p_3). \quad (4)$$

Note that we now have four equations to solve for four unknowns. This should be possible to do.

Here, it requires only a few more steps to solve for p_1 .

$$\begin{aligned} p_2 &= 0.8 - 0.6 = 0.2 \\ p_1 &= (p_1 + 0.2)0.6. \\ p_1 &= 0.3. \end{aligned}$$

There are usually other “easier” solutions. However, they are not as systematic as the solution above.

For example, one can start with

$$P(C \cup D) = P(C) + P(D) - P(C \cap D).$$

Forcing the events C and D to be independent means we must have $P(C \cap D) = P(C)P(D)$.

So,

$$P(C \cup D) = P(C) + P(D) - P(C)P(D).$$

Plugging in the provided values, we have

$$0.8 = P(C) + 0.6 - P(C)0.6.$$

This gives

$$P(C) = 0.5.$$

Therefore,

$$P(C \cap D) = P(C)P(D) = 0.5 \times 0.6 = \mathbf{0.3}.$$

EES 315: In-Class Exercise # 12

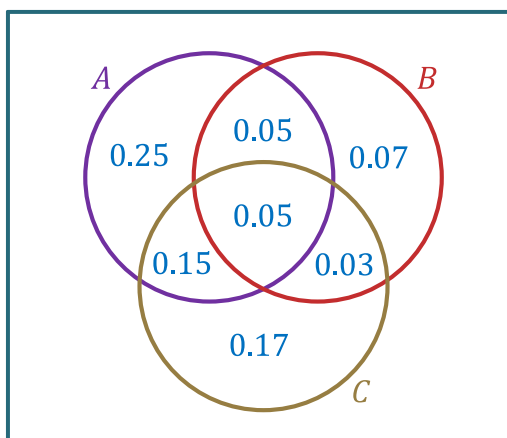
Instructions

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Date: 9 / 10 / 2020			
Name			ID <small>(last 3 digits)</small>

(1) Consider three events A , B , and C . Suppose

$$\begin{aligned}
 P(A^c \cap B \cap C) &= 0.03, & P(A \cap B^c \cap C) &= 0.15, & P(A \cap B \cap C^c) &= 0.05, \\
 P(A^c \cap B^c \cap C) &= 0.17, & P(A^c \cap B \cap C^c) &= 0.07, & P(A \cap B^c \cap C^c) &= 0.25, \text{ and} \\
 P(A \cap B \cap C) &= 0.05
 \end{aligned}$$



a. Are A , B , and C pairwise independent?

$$\begin{aligned}
 P(A) &= 0.25 + 0.15 + 0.05 + 0.05 = 0.5 \\
 P(B) &= 0.05 + 0.05 + 0.07 + 0.03 = 0.2 \\
 P(C) &= 0.05 + 0.15 + 0.03 + 0.17 = 0.4
 \end{aligned}$$

$$\begin{aligned}
 P(A \cap B) &= 0.05 + 0.05 = 0.1 \\
 P(A \cap C) &= 0.15 + 0.05 = 0.2 \\
 P(B \cap C) &= 0.05 + 0.03 = 0.08
 \end{aligned}$$

Checking pairwise independence for three events requires three conditions:

$$P(A \cap B) = P(A)P(B) \qquad P(A \cap C) = P(A)P(C) \qquad P(B \cap C) = P(B)P(C)$$

All three conditions are satisfied. Therefore, **yes**, the three events are pairwise independent.

b. Are A , B , and C independent?

Checking independence for three events requires four conditions. The first three conditions are the same as those for the pairwise independence which we have already checked in the previous part. Therefore, we only need to check the last condition:

$$P(A \cap B \cap C) \stackrel{?}{=} P(A)P(B)P(C).$$

Here, $P(A \cap B \cap C) = 0.05$. However, $P(A)P(B)P(C) = 0.5 \times 0.2 \times 0.4 = 0.04$. Therefore, the last condition fails. So, **no**, the three events are not independent.

EES 315: In-Class Exercise # 13

Instructions

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Date: 14 / 10 / 2020			
Name			ID <small>(last 3 digits)</small>

1) [Digital Communications] A certain binary-symmetric channel has a crossover probability (bit-error rate) of $p = 0.4$. Assume bit errors occur independently. Your answers for parts (a) and (b) should be of the form X.XXXX.

a) Suppose we input bit sequence “1010101” into this channel.

The BSC is used $n = 7$ times (this is the number of bits in both the input and the output sequences).

i) What is the probability that the output is “1000001”?

1010101
↓ ↓ ↓ ↓ ↓ ↓ ↓

This is similar to Example 6.54 in the lecture notes. However, note that p here is the probability of bit error. So, “success” here corresponds to the case that we have bit error; and “failure” corresponds to the case that the bit is unchanged when travels across the channel.

$$(1 - p) \times (1 - p) \times p \times (1 - p) \times p \times (1 - p) \times (1 - p) = (1 - p)^5 \times p^2 = 0.6^5 \times 0.4^2 = \frac{972}{78125} \approx 0.0124$$

$k = 4$

ii) What is the probability that exactly 4 bits are in error at the channel output?

See [6.55] and Example 6.60.a.i.

$$\binom{n}{k} p^k (1 - p)^{n-k} = \binom{7}{4} 0.4^4 0.6^3 = 35 \times 0.4^4 0.6^3 = \frac{3024}{15625} \approx 0.1935$$

iii) What is the probability that there is at least one bit error at the channel output?

First, we consider the opposite case: the probability that there is no bit error is

$$\binom{n}{0} p^0 (1 - p)^{n-0} = \binom{7}{0} 0.4^0 0.6^7 = 0.6^7.$$

Therefore, the probability that there is at least one bit error at the channel output is

$$1 - 0.6^7 \approx 0.9720.$$

b) Suppose we keep inputting bits into this channel. What is the probability that the **first** bit error at the output occurs on the fourth bit?

See Example 6.60.b.

Because the first bit error is on the fourth bit, this means the first three bits are unchanged across the channel and the fourth bit must be in error.

$$(1 - p) \times (1 - p) \times (1 - p) \times p = (1 - p)^3 \times p = 0.6^3 \times 0.4 = 0.0864$$

EES 315: In-Class Exercise # 14

Instructions

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 - (b) Hardcopy submission
4. **Do not panic.**

Date: 21 / 10 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider a random experiment in which you roll a six-sided fair dice (whose faces are numbered 1-6). We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega \quad \text{and} \quad Y(\omega) = (-1)^\omega.$$

- a. Find $P[X = 1]$.

$X(\omega) = 1$ when $\omega = 1$. Therefore, $P[X = 1] = P(\{1\}) = \frac{1}{6}$.

- b. Find $P[Y = 1]$.

$Y(\omega) = 1$ when $\omega = 2, 4, 6$.

Therefore, $P[Y = 1] = P(\{2, 4, 6\}) = \frac{3}{6} = \frac{1}{2}$.

2. Consider a random experiment in which you roll a 10-sided fair dice (whose faces are numbered 0–9). Define a random variable Z from the outcomes of this experiment by

$$Z(\omega) = (\omega - 5)^2.$$

- a. Find $P[Z = 25]$.

$Z(\omega) = 25$ when $(\omega - 5)^2 = 25$.

$\omega = 5 \pm 5 = 0$ or 10 .

Not in Ω .

Therefore, $P[Z = 25] = P(\{0\}) = \frac{1}{10}$.



$\Omega = \{0, 1, 2, \dots, 9\}$

- b. Find $P[Z > 5]$.

Method 1:

$Z(\omega) > 5$ when $(\omega - 5)^2 > 5$.

$\omega > 5 + \sqrt{5}$ or $\omega < 5 - \sqrt{5}$
 ≈ 7.2361 ≈ 2.7639

$\omega = 8$ or 9 $\omega = 0, 1,$ or 2

Therefore, $P[Z > 5] = P(\{0, 1, 2, 8, 9\}) = \frac{5}{10} = \frac{1}{2}$.

Method 2: Because Ω is not large, it is possible to find $Z(\omega)$ for all ω .

ω	$\omega - 6$	$(\omega - 6)^2$	
0	-5	25	
1	-4	16	> 5
2	-3	9	
3	-2	4	
4	-1	1	
5	0	0	
6	1	1	
7	2	4	
8	3	9	
9	4	16	> 5

EES 315: In-Class Exercise # 15

Instructions

1. Work alone or in a group of no more than three students. **The group cannot be the same as any of your former groups after the midterm.**
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4. **Do not panic.**

Date: 28 / 10 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider a random experiment in which you roll a six-sided fair dice (whose faces are numbered 1-6). We define the following random variable from the outcomes of this experiment:

$$Y(\omega) = (-1)^\omega.$$

- a. Find all possible values of the random variable Y .

ω	1	2	3	4	5	6
$Y(\omega)$	-1	1	-1	1	-1	1

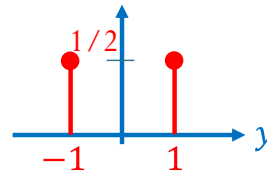
So, there are two possible values: **-1 and 1.**

- b. Plot its probability mass function $p_Y(y)$. (Recall that we use stem plot for pmf.)

$Y(\omega) = 1$ when $\omega = 2,4,6$. Therefore, $P[Y = 1] = P(\{2,4,6\}) = \frac{3}{6} = \frac{1}{2}$. (same as in Exc. 14)

$Y(\omega) = -1$ when $\omega = 1,3,5$. Therefore, $P[Y = -1] = P(\{1,3,5\}) = \frac{3}{6} = \frac{1}{2}$.

$$p_Y(y) \equiv P[Y = y] = \begin{cases} \frac{1}{2}, & y = -1, 1, \\ 0, & \text{otherwise.} \end{cases}$$



- c. Find $P[Y > -1]$.

We consider the two possible values of Y . Only “1” satisfies the condition “ > -1 ”.

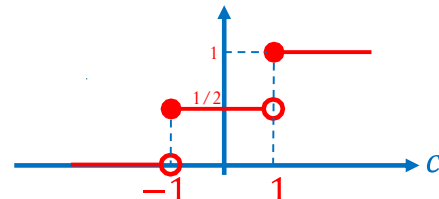
Therefore, $P[Y > -1] = p_X(1) = \frac{1}{2}$.

- d. Find $P[Y \leq 1.0001]$.

Both “-1” and “1” satisfy the condition “ ≤ 1.0001 ”. Therefore, $P[Y \leq 1.0001] = p_X(-1) + p_X(1) = 1$.

- e. (Optional) Plot $g(c) = P[Y \leq c]$ for all values of c between -2 and 2. (c may not be an integer.)

This function is exactly the same as the cdf except that the argument is c instead of the usual y . In particular, $g(c) = F_Y(c)$.



EES 315: In-Class Exercise # 16

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1. Work alone or in a group of no more than three students. **The group cannot be the same as any of your former groups after the midterm.**
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 - (b) Hardcopy submission
4. **Do not panic.**

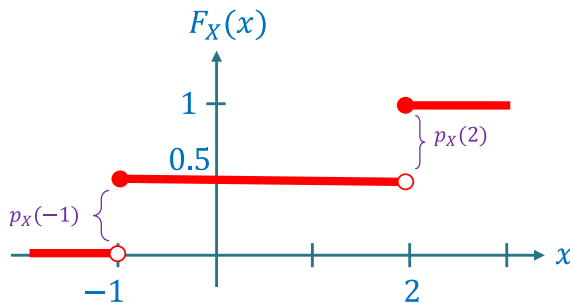
Date: 30 / 10 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider a random variable X whose pmf is given by $p_X(x) = \begin{cases} c, & x = -1, 2, \\ 0, & \text{otherwise.} \end{cases}$

a. Find the constant c .

$$“\Sigma = 1” \Rightarrow p_X(-1) + p_X(2) = 1 \Rightarrow c + c = 1 \Rightarrow c = 0.5$$

b. Plot the cdf of this random variable.



Recall that the cdf can be derived from the pmf by using the $p_X(x)$ as the jump amount at x .

2. Consider a random variable X whose cdf is given by

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.2, & 0 \leq x < 2, \\ 0.6, & 2 \leq x < 4, \\ 1, & x \geq 4. \end{cases}$$

↘ At $x = -1$, there is a jump of size 0.2.
↘ At $x = 2$, there is a jump of size 0.4.
↘ At $x = 4$, there is a jump of size 0.4.

a. Find $P[X \leq 3]$.

By definition, $P[X \leq 3] = F_X(3)$. Because $2 \leq 3 < 4$, we have $F_X(3) = 0.6$.

b. Find $P[X > 3]$.

Because $[X > 3]$ and $[X \leq 3]$ are opposite (complementary) events, we know that

$$P[X > 3] = 1 - P[X \leq 3] = 1 - 0.6 = 0.4.$$

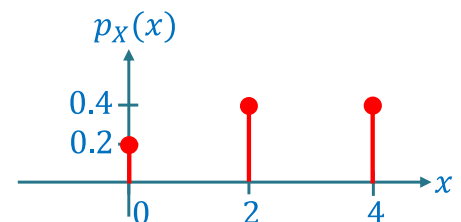
c. Plot the pmf of X .

For discrete RV, the pmf can be derived from the jump amounts in the cdf plot.

Here, the jumps in the cdf happen three times: at $x = 0$, $x = 2$, and $x = 4$.

The jump amounts are 0.2, 0.4, and 0.4, respectively.

$$\text{Therefore, } p_X(x) = \begin{cases} 0.2, & x = 0, \\ 0.4, & x = 2, \\ 0.4, & x = 4, \\ 0, & \text{otherwise.} \end{cases}$$



EES 315: In-Class Exercise # 17

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 - (b) Hardcopy submission
4. **Do not panic.**

Date: 4 / 11 / 2020			
Name			ID <small>(last 3 digits)</small>

Consider the random variable specified in each part below.

- i) Write down its (minimal) support.
- ii) Find $P[X = 1]$. Your answer should be of the form 0.XXXX.

	(minimal) support	$P[X = 1]$
$X \sim \mathcal{U}(\{-2, -1, 3\})$ $X \sim \text{Uniform}(S)$	The minimal support of a uniform RV is the set S being specified. Here, $S = \{-2, -1, 3\}$.	$p_X(x) = \begin{cases} \frac{1}{ S }, & x \in S, \\ 0, & \text{otherwise.} \end{cases}$ Here, $ S = 3$. Because $1 \notin S$, we have $P[X = 0] = p_X(1) = 0 = \mathbf{0.0000}$.
$X \sim \text{Bernoulli}(0.2)$ $X \sim \text{Bernoulli}(p)$	The (minimal) support of any Bernoulli RV is $\{0, 1\}$.	$p_X(x) = \begin{cases} 1 - p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$ Here, $p = 0.2$. $P[X = 1] = p_X(1) = p = 0.2 = \mathbf{0.2000}$.
$X \sim \mathcal{B}\left(3, \frac{1}{4}\right)$ $X \sim \text{Binomial}(n, p)$	The (minimal) support of a Binomial RV is $\{0, 1, \dots, n\}$. Here, $n = 3$. Therefore, the (minimal) support is $\{0, 1, 2, 3\}$.	$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$ Here, $n = 3$ and $p = \frac{1}{4}$. $p_X(x) = \begin{cases} \binom{3}{x} \left(\frac{1}{4}\right)^x \left(1 - \frac{1}{4}\right)^{3-x}, & x = 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$ Therefore, $P[X = 1] = p_X(1) = \binom{3}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{3-1}$ $= \left(\frac{3}{4}\right)^3 = \frac{27}{64} \approx \mathbf{0.4219}$.

EES 315: In-Class Exercise # 18

Instructions

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4. **Do not panic.**

Date: 6 / 11 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider the random variable specified in each part below.
 - (a) Write down its (minimal) support.
 - (b) Find $P[X \leq \pi]$. Your answer should be of the form 0.XXXX.

	(minimal) support	P[X ≤ π]
$X \sim \mathcal{G}_0(0.4)$	The minimal support of any Geometric ₀ RV is {0,1,2, ...} .	$p_X(x) = \begin{cases} p(1-p)^x, & x = 0,1,2, \dots, \\ 0, & \text{otherwise.} \end{cases}$ <p>The possible values of X are 0,1,2, Among these, only 0,1,2,3 satisfies the condition “$\leq \pi$”.</p> <p>Therefore,</p> $P[X \leq \pi] = p_X(0) + p_X(1) + p_X(2) + p_X(3)$ $= p(1-p)^0 + p(1-p)^1 + p(1-p)^2 + p(1-p)^3$ $= 0.4(1 + 0.6 + 0.6^2 + 0.6^3) \approx \mathbf{0.8704}.$
$X \sim \mathcal{P}(0.4)$	The minimal support of any Poisson RV is {0,1,2, ...} .	$p_X(x) = \begin{cases} e^{-\alpha} \frac{\alpha^x}{x!}, & x = 0,1,2, \dots, \\ 0, & \text{otherwise.} \end{cases}$ <p>The possible values of X are 0,1,2, Among these, only 0,1,2,3 satisfies the condition “$\leq \pi$”.</p> <p>Therefore,</p> $P[X \leq \pi] = p_X(0) + p_X(1) + p_X(2) + p_X(3)$ $= e^{-0.4} \frac{0.4^0}{0!} + e^{-0.4} \frac{0.4^1}{1!} + e^{-0.4} \frac{0.4^2}{2!} + e^{-0.4} \frac{0.4^3}{3!}$ $\approx \mathbf{0.9992}.$

2. [ENRPa] A certain binary-symmetric communication channel has a crossover probability (bit-error rate) of 0.4. Assume bit errors occur independently. For each of the random variables defined below, indicate the **name and the parameter(s)** of the family it belongs to.

Random Variable	Family
Consider the situation when 7 bits are input into this channel. Let N = the number of bit errors at the channel output.	$\mathcal{B}(7,0.4)$
Consider the situation when a sequence of bits $b_1 b_2 b_3 \dots$ are input into this channel. Let L = the index of the bit that corresponds to the first bit error at the channel output.	$\mathcal{G}_1(0.4)$

EES 315: In-Class Exercise # 19

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4. **Do not panic.**

Date: 11 / 11 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Find the expected value of the random variable X defined in each part below:

(a)
$$p_X(x) = \begin{cases} cx^2, & x \in \{-1, 1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$

First, we need to solve for the value of the unknown constant c .
To be a pmf, we need " $\Sigma = 1$ ". So,

$$\begin{aligned} p_X(-1) + p_X(1) + p_X(2) &= 1 \\ c(-1)^2 + c(1)^2 + c(2)^2 &= 1 \\ c &= \frac{1}{6} \end{aligned}$$

x	$p_X(x)$
-1	$c = \frac{1}{6}$
1	$c = \frac{1}{6}$
2	$4c = 4 \times \frac{1}{6} = \frac{2}{3}$

Therefore,

$$\begin{aligned} \mathbb{E}X &= \sum_x xp_X(x) \\ &= ((-1) \times p_X(-1)) + (1 \times p_X(1)) + (2 \times p_X(2)) \\ &= \left((-1) \times \frac{1}{6}\right) + \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{2}{3}\right) = \frac{4}{3} \approx 1.33. \end{aligned}$$

(b)
$$p_X(x) = \begin{cases} 0.4, & x = -1, 1, \\ c, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

" $\Sigma = 1$ ":
$$\begin{aligned} p_X(-1) + p_X(1) + p_X(2) &= 1 \\ 0.4 + 0.4 + c &= 1 \\ c &= 0.2. \end{aligned}$$

x	$p_X(x)$
-1	0.4
1	0.4
2	$c = 0.2$

$$\mathbb{E}X = \sum_x xp_X(x) = ((-1) \times 0.4) + (1 \times 0.4) + (2 \times 0.2) = 0.4.$$

(c)
$$F_X(x) = \begin{cases} 0, & x < -1, \\ 0.3, & -1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

This cdf has two jumps; one is @ $x = -1$ and another one is @ $x = 2$.

The jump sizes are 0.4 and 0.6, respectively.

x	$p_X(x)$
-1	0.3
2	0.7

$$\mathbb{E}X = \sum_x xp_X(x) = ((-1) \times 0.3) + (2 \times 0.7) = 1.1.$$

EES 315: In-Class Exercise # 20

Instructions

1. Work alone or in a group of no more than three students. **The group cannot be the same as any of your former groups after the midterm.**
2. Only one submission is needed for each group.
3. You have two choices for submission:
 - (a) Online submission via Google Classroom
 - PDF only.
 - Only for those who can directly work on the posted files using devices with pen input.
 - Paper size should be the same as the posted file.
 - No scanned work, photos, or screen capture.
 - **Your file name should start with the 10-digit student ID of one member.**
(You may add the IDs of other members, exercise #, or other information as well.)
 - (b) Hardcopy submission
4. **Do not panic.**

Date: 13 / 11 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Continue from the previous exercise.

Find $\mathbb{E}[X^2]$ and $\text{Var}[X]$ for the random variable X defined in each part below:

$p_X(x)$	$\mathbb{E}[X]$	$\mathbb{E}[X^2]$	$\text{Var}[X]$								
$p_X(x) = \begin{cases} \frac{1}{6}x^2, & x \in \{-1, 1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$ <table border="1" style="margin-left: auto; margin-right: auto; border-collapse: collapse;"> <thead> <tr> <th style="padding: 2px 5px;">x</th> <th style="padding: 2px 5px;">$p_X(x)$</th> </tr> </thead> <tbody> <tr> <td style="padding: 2px 5px;">-1</td> <td style="padding: 2px 5px;">$\frac{1}{6}$</td> </tr> <tr> <td style="padding: 2px 5px;">1</td> <td style="padding: 2px 5px;">$\frac{1}{6}$</td> </tr> <tr> <td style="padding: 2px 5px;">2</td> <td style="padding: 2px 5px;">$\frac{2}{3}$</td> </tr> </tbody> </table>	x	$p_X(x)$	-1	$\frac{1}{6}$	1	$\frac{1}{6}$	2	$\frac{2}{3}$	$\frac{4}{3}$	$= \sum_x x^2 p_X(x)$ $= \left((-1)^2 \times \frac{1}{6}\right) + \left(1^2 \times \frac{1}{6}\right) + \left(2^2 \times \frac{2}{3}\right)$ $= \frac{1}{6} + \frac{1}{6} + \frac{8}{3} = \frac{9}{3} = 3.$	$= \mathbb{E}[X^2] - (\mathbb{E}X)^2$ $= 3 - \left(\frac{4}{3}\right)^2 = \frac{11}{9}$ ≈ 1.2222
x	$p_X(x)$										
-1	$\frac{1}{6}$										
1	$\frac{1}{6}$										
2	$\frac{2}{3}$										
$p_X(x) = \begin{cases} 0.4, & x = -1, 1, \\ 0.2, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$ <table border="1" style="margin-left: auto; margin-right: auto; border-collapse: collapse;"> <thead> <tr> <th style="padding: 2px 5px;">x</th> <th style="padding: 2px 5px;">$p_X(x)$</th> </tr> </thead> <tbody> <tr> <td style="padding: 2px 5px;">-1</td> <td style="padding: 2px 5px;">0.4</td> </tr> <tr> <td style="padding: 2px 5px;">1</td> <td style="padding: 2px 5px;">0.4</td> </tr> <tr> <td style="padding: 2px 5px;">2</td> <td style="padding: 2px 5px;">0.2</td> </tr> </tbody> </table>	x	$p_X(x)$	-1	0.4	1	0.4	2	0.2	0.4	$= \sum_x x^2 p_X(x)$ $= \left((-1)^2 \times 0.4\right) + \left(1^2 \times 0.4\right) + \left(2^2 \times 0.2\right)$ $= 0.4 + 0.4 + 0.8$ $= 1.6 = \frac{8}{5}$	$= \mathbb{E}[X^2] - (\mathbb{E}X)^2$ $= 1.6 - (0.4)^2$ $= 1.44 = \frac{36}{25}$
x	$p_X(x)$										
-1	0.4										
1	0.4										
2	0.2										

EES 315: In-Class Exercise # 21

Instructions

1. Work alone or in a group of no more than three students. **The group cannot be the same as any of your former groups after the midterm.**
2. Only one submission is needed for each group.
3. You have two choices for submission:
 - (a) Online submission via Google Classroom
 - PDF only.
 - Only for those who can directly work on the posted files using devices with pen input.
 - Paper size should be the same as the posted file.
 - No scanned work, photos, or screen capture.
 - **Your file name should start with the 10-digit student ID of one member.**
(You may add the IDs of other members, exercise #, or other information as well.)
 - (b) Hardcopy submission
4. **Do not panic.**

Date: 17 / 11 / 2020			
Name			ID <small>(last 3 digits)</small>

In this question, we consider two distributions for a random variable X .

In part (a), which corresponds to the second column in the table below, X is a **discrete** random variable with its pmf specified in the first row.

In part (b), which corresponds to the third column, X is a **continuous** random variable with its pdf specified in the first row.

	$p_X(x) = \begin{cases} cx^3, & x \in \{1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$	$f_X(x) = \begin{cases} cx^3, & x \in [1, 2), \\ 0, & \text{otherwise.} \end{cases}$						
Find c	<p>“$\Sigma = 1$”:</p> $\begin{aligned} p_X(1) + p_X(2) &= 1 \\ c(1)^3 + c(2)^3 &= 1 \\ 9c &= 1 \\ c &= \frac{1}{9}. \end{aligned}$ <table border="1" style="margin-left: auto; margin-right: auto; border-collapse: collapse;"> <thead> <tr> <th style="padding: 2px;">x</th> <th style="padding: 2px;">$p_X(x)$</th> </tr> </thead> <tbody> <tr> <td style="text-align: center; padding: 2px;">1</td> <td style="padding: 2px;">$c(1)^3 = c = \frac{1}{9}$</td> </tr> <tr> <td style="text-align: center; padding: 2px;">2</td> <td style="padding: 2px;">$c(2)^3 = 8c = \frac{8}{9}$</td> </tr> </tbody> </table>	x	$p_X(x)$	1	$c(1)^3 = c = \frac{1}{9}$	2	$c(2)^3 = 8c = \frac{8}{9}$	<p>We need “$\int = 1$”.</p> $\begin{aligned} \int_{-\infty}^{\infty} f_X(x) dx &= \int_1^2 f_X(x) dx \\ &= \int_1^2 cx^3 dx = \frac{cx^4}{4} \Big _1^2 \\ &= \frac{c}{4} (2^4 - 1^4) = \frac{15}{4}c \end{aligned}$ <p>Therefore,</p> $\begin{aligned} \frac{15}{4}c &= 1 \\ c &= \frac{4}{15}. \end{aligned}$
x	$p_X(x)$							
1	$c(1)^3 = c = \frac{1}{9}$							
2	$c(2)^3 = 8c = \frac{8}{9}$							
Find $P[0 < X \leq 1]$	<p>The possible values of this RV are 1 and 2. Among these, only “1” satisfies the condition. Therefore,</p> $\begin{aligned} P[0 < X \leq 1] &= P[X = 1] \\ &= p_X(1) = \frac{1}{9}. \end{aligned}$	<p>$f_X(0) = 0$</p> $\begin{aligned} P[0 < X \leq 1] &= \int_0^1 f_X(x) dx = \int_0^1 dx \quad \text{inside this interval} \\ &= \frac{cx^4}{4} \Big _0^1 = \frac{c}{4} (1^4 - 0^4) \\ &= \frac{c}{4} = \frac{1}{15} = 0 \end{aligned}$						

EES 315: In-Class Exercise # 22

Instructions

1. Work alone or in a group of no more than three students. **The group cannot be the same as any of your former groups after the midterm.**
2. Only one submission is needed for each group.
3. You have two choices for submission:
 - (a) Online submission via Google Classroom
 - PDF only.
 - Only for those who can directly work on the posted files using devices with pen input.
 - Paper size should be the same as the posted file.
 - No scanned work, photos, or screen capture.
 - **Your file name should start with the 10-digit student ID of one member.**
(You may add the IDs of other members, exercise #, or other information as well.)
 - (b) Hardcopy submission
4. **Do not panic.**

Date: 25 / 11 / 2020			
Name			ID <small>(last 3 digits)</small>

Continue from the previous exercise. We consider two distributions for a random variable X .

In part (a), which corresponds to the second column in the table below, X is a **discrete** random variable with its pmf specified in the first row.

In part (b), which corresponds to the third column, X is a **continuous** random variable with its pdf specified in the first row.

	$p_X(x) = \begin{cases} \frac{1}{9}x^3, & x \in \{1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$	$f_X(x) = \begin{cases} \frac{4}{15}x^3, & x \in [1, 2], \\ 0, & \text{otherwise.} \end{cases}$
Find $\mathbb{E}X$	$\begin{aligned} \mathbb{E}X &= \sum_x x p_X(x) = \sum_{x \in \{1, 2\}} x \left(\frac{1}{9}x^3 \right) \\ &= \frac{1}{9} \sum_{x \in \{1, 2\}} x^4 = \frac{1}{9} (1^4 + 2^4) \\ &= \frac{17}{9} \approx 1.8889 \end{aligned}$	$\begin{aligned} \mathbb{E}X &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_1^2 x \left(\frac{4}{15}x^3 \right) dx = \frac{4}{15} \int_1^2 x^4 dx \\ &= \frac{4}{15} \frac{x^5}{5} \Big _1^2 = \frac{4}{75} (2^5 - 1^5) \\ &= \frac{124}{75} \approx 1.6533 \end{aligned}$
Find $\mathbb{E}[X^2]$	$\begin{aligned} \mathbb{E}[X^2] &= \sum_x x^2 p_X(x) = \sum_x x^2 \left(\frac{1}{9}x^3 \right) \\ &= \frac{1}{9} \sum_{x \in \{1, 2\}} x^5 = \frac{1}{9} (1^5 + 2^5) \\ &= \frac{33}{9} = \frac{11}{3} \approx 3.6667 \end{aligned}$	$\begin{aligned} \mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_1^2 x^2 \left(\frac{4}{15}x^3 \right) dx = \frac{4}{15} \int_1^2 x^5 dx \\ &= \frac{4}{15} \frac{x^6}{6} \Big _1^2 = \frac{2}{45} (2^6 - 1^6) \\ &= \frac{2 \times 63}{45} = \frac{14}{5} = 2.8 \end{aligned}$
Find $\text{Var}[X]$	$\begin{aligned} \text{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 \\ &= \frac{11}{3} - \left(\frac{17}{9} \right)^2 \\ &= \frac{8}{81} \approx 0.0988 \end{aligned}$	$\begin{aligned} \text{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 \\ &= \frac{14}{5} - \left(\frac{124}{75} \right)^2 \\ &= \frac{374}{5625} \approx 0.0665 \end{aligned}$

<p>(optional) Find $\text{Var}[X^2]$</p>	$\begin{aligned} \mathbb{E}[X^4] &= \sum_x x^4 p_X(x) = \sum_x x^4 \left(\frac{1}{9}x^3\right) \\ &= \frac{1}{9} \sum_{x \in \{1,2\}} x^7 = \frac{1}{9}(1^7 + 2^7) \\ &= \frac{129}{9} \approx 14.3333 \\ \text{Var}[X^2] &= \mathbb{E}[X^4] - (\mathbb{E}[X^2])^2 \\ &= \frac{129}{9} - \left(\frac{11}{3}\right)^2 = \frac{8}{9} \approx 0.8889 \end{aligned}$	$\begin{aligned} \mathbb{E}[X^4] &= \int_{-\infty}^{\infty} x^4 f_X(x) dx \\ &= \int_1^2 x^4 \left(\frac{4}{15}x^3\right) dx = \frac{4}{15} \int_1^2 x^7 dx \\ &= \frac{4}{15} \frac{x^8}{8} \Big _1^2 = \frac{1}{15 \times 2} (2^8 - 1^8) \\ &= \frac{255}{15 \times 2} = \frac{17}{2} = 8.5 \\ \text{Var}[X^2] &= \mathbb{E}[X^4] - (\mathbb{E}[X^2])^2 \\ &= \frac{17}{2} - \left(\frac{14}{5}\right)^2 = \frac{33}{50} \approx 0.66 \end{aligned}$
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$\text{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2$
 Set $Y = X^2$
 $\text{Var}[X^2] = \mathbb{E}[X^4] - (\mathbb{E}[X^2])^2$