Sirindhorn International Institute of Technology

Thammasat University at Rangsit

School of Information, Computer and Communication Technology

ECS 455: Problem Set 4

Semester/Year: 2/2016

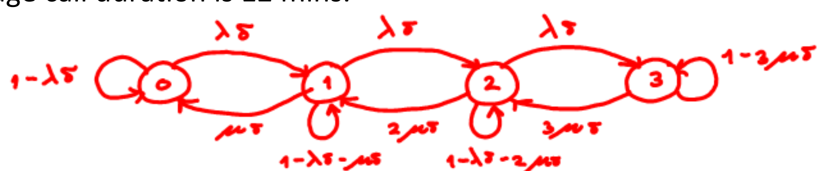
Course Title: Mobile Communications
Instructor: Asst. Prof. Dr. Prapun Suksompong (prapun@siit.tu.ac.th)
Course Web Site: http://www2.siit.tu.ac.th/prapun/ecs455/

Due date: Not Due

Questions

1. Consider a system which has 3 channels. We would like to find the blocking probability via the Markov chain method. For each of the following models, **draw the Markov chain** via discrete time approximation. Find (1) the **steady-state probabilities** and (2) the long-term **call blocking probability**.
    a. ***Erlang B*** model: Assume that the *total* call request rate is 10 calls per hour and the average call duration is 12 mins.

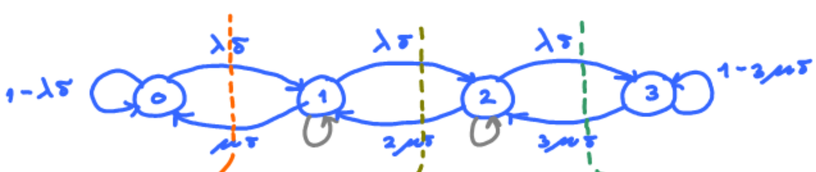
Markov Chain:



$$A = \frac{\lambda}{\mu} = \lambda \times \frac{1}{\mu}$$

$$= \left(\cancel{10}^{\,2} \frac{\text{calls}}{\text{hour}} \times \frac{1 \text{ hour}}{\cancel{60}_{5} \text{ mins}}\right) \times \left(\cancel{12} \text{ mins}\right)$$

$$= 2 \text{ Erlangs.}$$



Balance equations:

$p_0 \lambda\cancel{\delta} = p_1 \mu\cancel{\delta}$ $\qquad$ $p_1 \lambda\cancel{\delta} = p_2 2\mu\cancel{\delta}$ $\qquad$ $p_2 \lambda\cancel{\delta} = 3\mu\cancel{\delta} p_3$

$p_1 = A p_0$ $\qquad$ $p_2 = \frac{A}{2} p_1 = \frac{A^2}{2} p_0$ $\qquad$ $p_3 = \frac{A}{3} p_2 = \frac{A^3}{3!} p_0$

$$p_0 + p_1 + p_2 + p_3 = 1 \Rightarrow p_0 = \left(1 + A + \frac{A^2}{2} + \frac{A^3}{3}\right)^{-1} \Rightarrow \begin{cases} p_0 = \frac{3}{19}, \\ p_1 = \frac{6}{19}, \\ p_2 = \frac{6}{19} \\ p_3 = \frac{4}{19} \end{cases}$$

$\Downarrow$

call blocking probability $= \frac{4}{19} \approx 0.211$

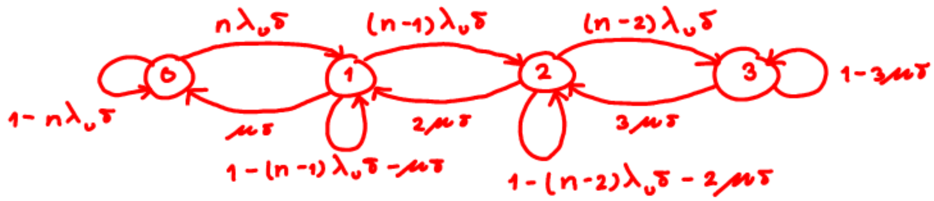
b. ***Engset*** model: Assume that there are 5 users. The call request rate for _each_ user is 2 calls per hour and the average call duration is 12 mins.

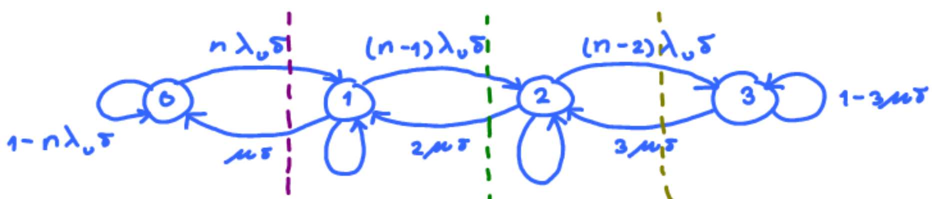
c. ***Engset*** model: Assume that there are 100 users. The call request rate for _each_ user is 0.1 calls per hour and the average call duration is 12 mins.

Observe that $\lambda_u \times n = \lambda$ in part (a).

So, $\lambda_u = \frac{\lambda}{n}$ and $A_u = \frac{A}{n} = \frac{2}{n}$ Erlangs.

Markov chain:





$p_0 n \lambda_u \cancel{\delta} = p_1 \mu \cancel{\delta}$ $\quad$ $p_1 (n-1)\lambda_u \cancel{\delta} = 2\mu\cancel{\delta} p_2$ $\quad$ $p_2 (n-2)\lambda_u \cancel{\delta} = p_3 3\mu\cancel{\delta}$

$p_1 = nA_u$ $\quad$ $p_2 = \frac{(n-1)}{2} A_u p_1 = \frac{n(n-1)}{2} A_u^2 p_0$ $\quad$ $p_3 = \frac{(n-2)}{3} A_u p_2 = \frac{n(n-1)(n-2)A_u^3 p_0}{3!}$

$$p_0 + p_1 + p_2 + p_3 = 1 \Rightarrow \begin{cases} p_0 = \frac{25}{131},\ p_1 = \frac{50}{131},\ p_2 = \frac{40}{131},\ p_3 = \frac{16}{131} & \text{when } n=5, \\ p_0 = 0.159,\ p_1 = 0.319,\ p_2 = 0.316,\ p_3 = 0.206 & \text{when } n=100. \end{cases}$$

As discussed in class, the call blocking probability is given by

$$\frac{(n-m)p_m}{\sum_{k=0}^{m} (n-k)p_k} = \frac{(n-3)p_3}{np_0 + (n-1)p_1 + (n-2)p_2 + (n-3)p_3}$$

$$= \begin{cases} \frac{32}{477} \approx 0.067 & \text{when } n=5 \\ 0.203 & \text{when } n=100 \end{cases}$$

close to the answer from Erlang B.
because there are many users.

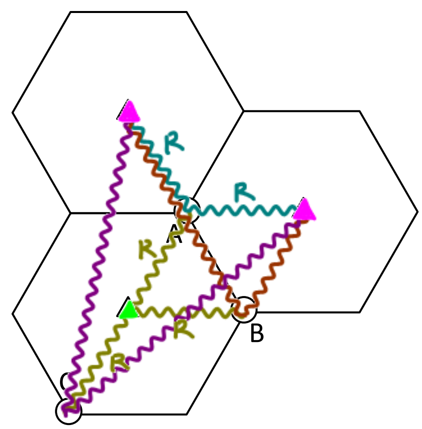
Extra Questions

Here are some extra questions for those who want more practice.

2. **[M2009/1Q2]** In this question, we consider the SIR value when the cluster size $N = 1$. Sectoring is not used. Suppose this cellular system has only three base stations. They are marked by triangles located at the centers of three cells in the figure below. Assume that they transmit the same power level.

User A (mobile station A), user B (mobile station B), and user C (mobile station C) are currently associated with the lower-left base station. The locations of these users are marked by the circles in the figure.

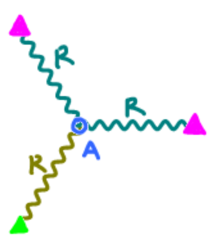
Assume a path loss exponent of $\gamma = 4$. Do not approximate distance values.



First, note that the distance from each user to the associated BS is R.

a. Find the signal-to-interference ratio (in dB) for user A.

Both distances from A to the interfering BS are also R.

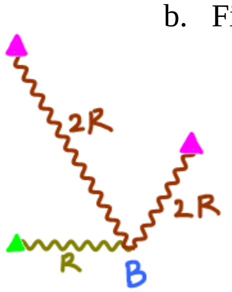


$$\text{So, SIR} = \frac{P_r}{\sum P_i} = \frac{1/R^\gamma}{\frac{1}{R^\gamma} + \frac{1}{R^\gamma}} = \frac{1}{2} = -3\,[\text{dB}]$$

Note that we have only two (not six) interferers in this problem.

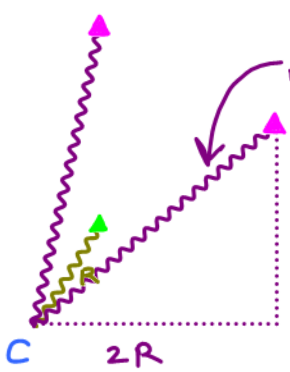
Remark: Note also that the signals from B and C will not interfere with the signal from A because they use different bands of frequency.

b. Find the signal-to-interference ratio (in dB) for user B.



$$\text{SIR} = \frac{\frac{1}{R^\gamma}}{\frac{1}{R^\gamma} + \frac{1}{(2r)^\gamma}} = \frac{1}{1 + \frac{1}{2^\gamma}} \underset{\gamma=4}{=} \frac{1}{1 + \frac{1}{16}} = -0.263\,[\text{dB}]$$

c. Find the signal-to-interference ratio (in dB) for user C.



$$\sqrt{2^2 + \sqrt{3}^2}\,R = \sqrt{7}\,R$$

$$\cancel{2} \times \frac{\sqrt{3}}{\cancel{2}} R$$

$$\text{SIR} = \frac{\frac{1}{R^\gamma}}{2 \times \frac{1}{(\sqrt{7}R)^\gamma}} = \frac{\sqrt{7}^\gamma}{2} \underset{\gamma=4}{=} \frac{49}{2} = 13.9\,[\text{dB}]$$

3. **[M2009/1Q5]** A function $\text{ErlangB}(m, A)$ is defined by

$$\text{ErlangB}(m, A) = \frac{\frac{A^m}{m!}}{\sum_{k=0}^{m} \frac{A^k}{k!}}$$

a. (2 pt) Compare $\text{ErlangB}(\overset{m}{1000}, \overset{A}{2000})$ and $\text{ErlangB}(\overset{m}{1000}, \overset{A}{2001})$. Which one is larger?

more Erlang
⇒ more traffic
⇒ larger P_b

b. (2 pt) Compare $\text{ErlangB}(\overset{m}{1000}, \overset{A}{2000})$ and $\text{ErlangB}(\overset{m}{1001}, \overset{A}{2000})$. Which one is larger?

less # channels
⇒ larger P_b

c. (1 pt) Suppose

$$\text{ErlangB}(1000, A_1) = \text{ErlangB}(2000, A_2) = \text{ErlangB}(3000, A_3) = 0.05$$

Let $A_4 = A_1 + A_2$.

Compare A_3 and A_4. Which one is larger? A_3

Think abuot the case when 3,000 channels are divided into two groups of sizes 1,000 and 2,000.
The trunking efficiency should be higher when all the 3,000 channels are in the same pool. Therefore, it should be able to support more traffic for the same blocking probability.

4. **[M2009/1Q4]** There are 1000 users subscribed to a cellular system. The call request rate for each user is 2 call requests per week. For each call, the average call duration is 1 min. If the system has only two channels and it is used to support the whole 1000 users, what is the blocking probability? $m = 2$

$\frac{1}{\mu} = 1$ min.

Under Erlang B model:

$$\lambda = 2 \times 1{,}000 \ \frac{\text{requests}}{\text{week}} \times \frac{1 \text{ week}}{7 \text{ days}} \times \frac{1 \text{ day}}{24 \text{ hrs.}} \times \frac{1 \text{ hr}}{60 \text{ min}}$$

$$A = \frac{\lambda}{\mu} = \frac{2 \times 1000}{7 \times 24 \times 60} = 0.198$$

$m = 2$

Erlang B formula

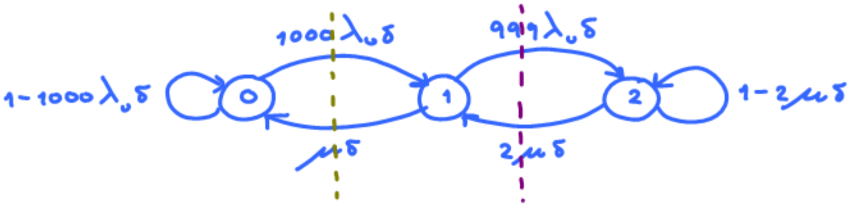
$$\rightarrow P_b = \frac{A^2/2}{1 + A + \frac{A^2}{2}} = \frac{625}{38677} \approx 0.0162$$

Under the Engset model

$$\lambda_u = 2\ \frac{\text{call}}{\text{week}} \times \frac{1\ \text{week}}{60\times24\times7\ \text{mins}}$$

$$\mu = 1\ \text{min}$$

$$A_u = \frac{\lambda_u}{\mu} = \frac{1}{5040}\ [\text{Erlangs}]$$



$$1000\times p_0\times\lambda_u\cancel{\delta} = p_1\mu\cancel{\delta}$$

$$999\lambda_u\cancel{\delta}\, p_1 = p_2\times 2\mu\cancel{\delta}$$

$$p_1 = 1000\frac{\lambda_u}{\mu}p_0 = 1000A_u p_0$$

$$p_2 = \frac{999}{2}\frac{\lambda_u}{\mu}p_1 = \frac{1000\times999}{2}A_u^2 p_0$$

$$p_0+p_1+p_2 = 1 \Rightarrow \left(1+\frac{100\cancel{0}}{504\cancel{0}}+\frac{5\cancel{00}\times999}{504\cancel{0}^2}\right)p_0 = 1$$

$$\left(504^2+50400+5\times999\right)p_0 = 504^2$$

$$(309411)\ p_0 = 254016$$

$$p_0 = \frac{28224}{34379} \approx 0.8210$$

$$p_1 = 1000A_u p_0 = \frac{5600}{34379} \approx 0.1629$$

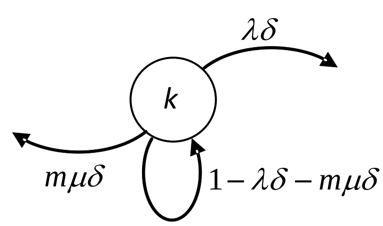
$$p_2 = \frac{999}{2}A_u p_1 = \frac{555}{34379} \approx 0.0161$$

Finally, $$P_b = \frac{(n-m)p_m}{\sum_{k=0}^{m}(n-k)p_k} = \frac{18463}{1145743} \approx 0.0161$$

close to the answer from the Erlang B model)

Remark:
1) In 2009, at the time of the midterm, the Engset model was still not covered. So, they know only the Erlang B model. Therefore, Erlang B was the only choice.
2) The number of users in the system is large. Therefore, it is reasonable to try to use the Erlang B model anyway knowing that the answer will be close to the answer from the Engset model.

5. Consider another modification of the M/M/m/m (Erlang B) system. (There are infinite users) Assume that there is a queue that can be used to hold all requested call which cannot be immediately assigned a channel. This is referred to as an M/M/m/∞ or simply M/M/m system. We will define state *k* as the state where there are *k* calls in the system. If $k \le m$, then all of these calls are ongoing. If $k > m$, then *m* of them are ongoing and *k*-*m* of them are waiting in the queue.
Assume that the total call request process is Poisson with rate λ and that the call durations are i.i.d. exponential random variables with expected value $1/\mu$.
Also assume that $\frac{\lambda}{\mu} < m$.
    a. **<u>Draw the Markov chain</u>** via discrete time approximation. Don't forget to indicate the transition probabilities (in terms of λ, μ, and δ) on the arrows.
    Hint: there are infinite number of states. The transition probabilities for state *k* which is < *m* are the same as in the M/M/m/m system. For $k \ge m$, the transition probabilities are given below:



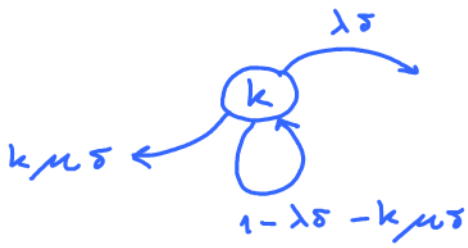
    Explain why the above transition probabilities make sense.
    b. Find the **<u>steady-state probabilities</u>** using balance equations
    c. Find the long-term **<u>delayed call probability</u>** (the probability that a call request occurs when all *m* channels are busy and thus has to wait).
    Hint: This will be a summation of many steady-state probabilities. When you simplify your answer, the final answer should be

$$\frac{A^m}{A^m + m!\left(1-\frac{A}{m}\right)\sum_{k=0}^{m-1}\frac{A^k}{k!}}.$$

M/M/m and the Erlang C Formula

(a) When k < m, nothing change from the M/M/m/m model .

We have



When $k \geqslant m$,

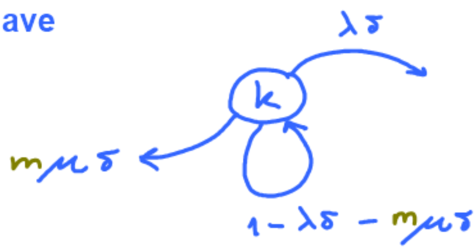
The call request rate is still λ.
All m channels are being used.
The difference is that now we have a queue for the new requests to wait.

In M/M/m/m, these requests are discarded and the calls are blocked.
Here, there are k-m requests waiting in the queue.

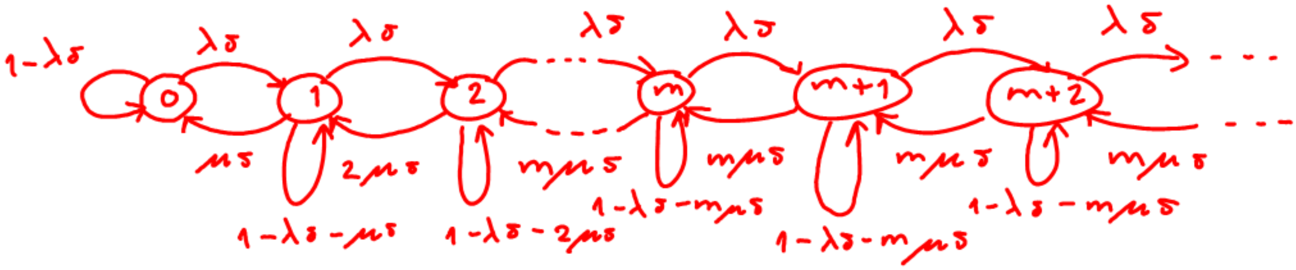
When there is one new call request, it will be added to the queue and hence the system moves from state k to k+1. Again, this new call request occurs with probability approximately $\lambda\delta$.

Although that state of the system is k, only m calls are ongoing. (The other k-m requests are waiting in the queue.) The probability that one of the m ongoing calls end is approximately $m\mu\delta$ not $k\mu\delta$.

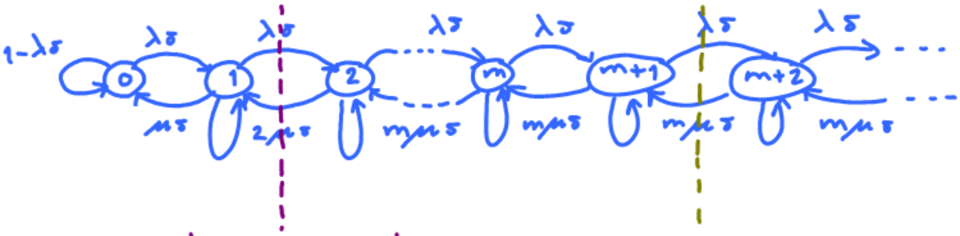
Therefore, we have



Hence, the Markov chain is



(b) steady-state probabilities



$$p_{k-1}\lambda\cancel{\delta} = p_k k\mu\cancel{\delta} \qquad\qquad p_{k-1}\lambda\cancel{\delta} = m\mu\cancel{\delta}\, p_k$$

$$p_k = \frac{A}{k} p_{k-1} \qquad\qquad p_k = \frac{A}{m} p_{k-1}$$

$$= \frac{A^k}{k!} p_0 \qquad\qquad \text{for } k \geq m$$

for $0 < k < m$

$$\Rightarrow p_m = \frac{A}{m} p_{m-1} = \frac{A}{m}\frac{A^{m-1}}{(m-1)!} p_0 = \frac{A^m}{m!} p_0$$

$$p_k = \left(\frac{A}{m}\right)^{k-m} \frac{A^m}{m!} p_0 = \frac{A^k}{m!\,(m^{k-m})} p_0$$

$$= \frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0 \quad \text{for } k \geq m.$$

$$\sum_{k=0}^{m} p_k = 1 \Rightarrow 1 = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \underbrace{\sum_{k=m}^{\infty} \frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0} = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \frac{A^m}{m!\left(1-\frac{A}{m}\right)} p_0$$

geometric series

$$= \begin{cases} \dfrac{m^m}{m!} \dfrac{\left(\frac{A}{m}\right)^m}{1-\frac{A}{m}} p_0 & \text{if } A < m \\ \infty & \text{if } A \geq m \end{cases}$$

$$\Rightarrow p_0 = \left(\left(\sum_{k=0}^{m-1} \frac{A^k}{k!}\right) + \frac{A^m}{m!\left(1-\frac{A}{m}\right)} \right)^{-1}$$

Therefore,

$$p_k = \begin{cases} \dfrac{A^k}{k!} p_0, & k < m \\ \dfrac{A^k}{m!\,(m^{k-m})} p_0, & k \geq m \end{cases}$$

(c) Delayed call probability

$$= \sum_{k=m}^{\infty} p_k = \frac{A^m}{m!\left(1-\frac{A}{m}\right)} p_0 = \frac{\dfrac{A^m}{m!\,(1-A/m)}}{\dfrac{A^m}{m!\left(1-\frac{A}{m}\right)} + \displaystyle\sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

$$= \frac{A^m}{A^m + m!\left(1-\frac{A}{m}\right) \displaystyle\sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

Remark: This formula is call the "Erlang C formula".

6. Consider the Engset model
    a. Show that the steady-state probabilities for the Engset model are given by
$$p_i = \frac{\binom{n}{i} A_u^i}{\sum_{k=0}^{m} \binom{n}{k} A_u^k} = \frac{\binom{n}{i} A_u^i}{z(m,n)},\ 0 \le i \le m,$$
    where $z(m,n) = \sum_{k=0}^{m} \binom{n}{k} A^k$.
    b. Express p_m (time congestion) in the form $p_m = 1 - \frac{z(m-c,n)}{z(m,n)}$.

       What is the value of c?

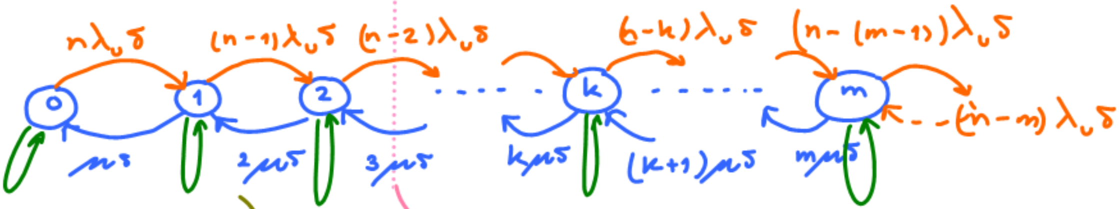
       Hint: c is an integer.
    c. Show that the blocked call probability is given by $P_b = \frac{(n-m)\binom{n}{m} A_u^m}{\sum_{k=0}^{m} (n-k)\binom{n}{k} A_u^k}$.
    d. The blocked call probability can be rewritten in the form $P_b = 1 - \frac{z(m-c_1, n-c_2)}{z(m-c_3, n-c_4)}$.

       Find c_1, c_2, c_3, c_4.

       Hint: c_1, c_2, c_3, c_4 are all integers.
    e. Suppose $m = n-1$. Simplify the expression for P_b.

       Hint: Your answer should be of the form $\left(g\left(A_u\right)\right)^m$ for some function *g* of *A_u*.

(a) steady-state probabilities for the Engset model



$$p_0 n\lambda_u \cancel{\delta} = p_1 \mu \cancel{\delta}$$

recall that $A_u = \dfrac{\lambda_u}{\mu}$

$$p_1 = nA_u p_0$$

$$p_1(n-1)\lambda_u\cancel{\delta} = p_2\, 2\mu\cancel{\delta}$$

$$p_2 = \frac{n-1}{2}A_u p_1 = \frac{(n-1)}{2} n A_u^2 p_0 = \binom{n}{2}A_u^2 p_0$$

$$p_2(n-2)\lambda_u\cancel{\delta} = p_3\, 3\mu\cancel{\delta}$$

$$p_3 = \frac{n-2}{3}A_u p_2 = \frac{n(n-1)(n-2)}{3\times 2}A_u^3 p_0 = \binom{n}{3}A_u^3 p_0$$

$$p_k = \binom{n}{k}A_u^k p_0 = \frac{\binom{n}{k}A_u^k}{\sum_{i=0}^{m}\binom{n}{i}A_u^i} = \frac{\binom{n}{k}A_u^k}{z(m,n)}$$

Note that z(m,n) is simply a normalizing factor

$$\sum_{k=0}^{m} p_k = 1 \Rightarrow \sum_{k=0}^{m}\binom{n}{k}A_u^k p_0 = 1 \Rightarrow p_0 = \frac{1}{\underbrace{\sum_{k=0}^{m}\binom{n}{k}A_u^k}_{z(m,n)}}$$

(b)

$$p_m = \frac{\binom{n}{m}A_u^m}{\sum_{k=0}^{m}\binom{n}{k}A_u^k} = \frac{\sum_{k=0}^{m}\binom{n}{k}A_u^k - \sum_{k=0}^{m-1}\binom{n}{k}A_u^k}{\sum_{k=0}^{m}\binom{n}{k}A_u^k}$$

$$= \frac{z(m,n) - z(m-1,n)}{z(m,n)} = 1 - \frac{z(m-1,n)}{z(m,n)}$$

Hence, $c = 1$

(c) In class, we show that

$$P_b = \frac{(n-m)p_m}{\sum_{k=0}^{m}(n-k)p_k} \overset{(a)}{=} \frac{(n-m)\dfrac{A_u^m\binom{n}{m}}{\cancel{z(m,n)}}}{\sum_{k=0}^{m}(n-k)\dfrac{A_u^k\binom{n}{k}}{\cancel{z(m,n)}}} = \frac{(n-m)\binom{n}{m}A_u^m}{\sum_{k=0}^{m}(n-k)\binom{n}{k}A_u^k}$$

(d) First, note that

$$(n-k)\times\binom{n}{k} = (n-k)\times\frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k-1)!} = \frac{n\times(n-1)!}{k!(n-1-k)!} = n\binom{n-1}{k}$$

Therefore, $$P_b = \frac{(n-m)\binom{n}{m}A_u^m}{\sum_{k=0}^{m}(n-k)\binom{n}{k}A_u^k} = \frac{\cancel{n}\binom{n-1}{m}A_u^m}{\sum_{k=0}^{m}\cancel{n}\binom{n-1}{k}A_u^k}$$

$$= \frac{\sum_{k=0}^{m}\binom{n-1}{k}A_u^k - \sum_{k=0}^{m-1}\binom{n-1}{k}A_u^k}{\sum_{k=0}^{m}\binom{n-1}{k}A_u^k}$$

$$= \frac{z(m,n-1) - z(m-1,n-1)}{z(m,n-1)} = 1 - \frac{z(m-1,n-1)}{z(m,n-1)}$$

Hence, $c_1 = c_2 = c_4 = 1$ and $c_3 = 0$.

(e) $$P_b = \frac{\binom{n-1}{m}A_u^m}{\sum_{k=0}^{m}\binom{n-1}{k}A_u^k} \overset{m=n-1}{=} \frac{\binom{m}{m}A_u^m}{\sum_{k=0}^{m}\binom{m}{k}A_u^k} = \frac{A_u^m}{(1+A_u)^m} = \left(\frac{A_u}{1+A_u}\right)^m$$