Q1 Friis Equation

Monday, November 12, 2012
9:34 PM

Under the free-space PL model, we have the Friis equation:

$$\frac{P_r}{P_t} = \left(\frac{\sqrt{G_{Tx} G_{Rx}}}{4\pi d} \lambda \right)^2 = \left(\frac{\sqrt{G_{Tx} G_{Rx}}}{4\pi d} \frac{c}{f} \right)^2$$

Therefore,
$$P_r = \underbrace{\left(\frac{\sqrt{G_{Tx} G_{Rx}}}{4\pi d} c \right) P_t}_{\text{fixed, for this question}} \frac{1}{f^2} \propto \frac{1}{f^2}$$

When the frequency f is changed from f_{old} to f_{new}, the relationship between the old received power and the new received power is given by

$$\frac{P_{r,new}}{P_{r,old}} = \frac{f_{old}^2}{f_{new}^2}$$

In dB, the change (gain) in power is

$$\underbrace{10 \log_{10} \frac{P_{r,new}}{P_{r,old}}}_{= P_{r,new}[dB] - P_{r,old}[dB]} = 10 \log_{10} \left(\frac{f_{old}}{f_{new}} \right)^2 = 20 \log_{10} \frac{f_{old}}{f_{new}}.$$

In this question, $f_{old} = 700$ MHz $= 0.7$ GHz and

$f_{new} = 1800$ MHz $= 1.8$ GHz.

Therefore,
$$P_{r,new}[dB] - P_{r,old}[dB] = 20 \log_{10} \frac{0.7}{1.8} = -8.2 \text{ dB}.$$

The negative result tells us that the received power when higher frequency is used would be lower, which is what we expect from the $\frac{1}{f^2}$ proportionality found earlier.

In conclusion, the loss in received power is 8.2 dB (about 87%)

Q2

Thursday, November 15, 2012
8:43 AM

Recall that $c = f\lambda$, which means $\lambda = \frac{c}{f}$

Here, $f = 0.9 \times 10^9$, 1.9×10^9, and 5.8×10^9 Hz.

Hence, $\lambda = 33.3$, 15.8 , and 5.17 cm

Q3

Thursday, November 15, 2012
9:13 AM

(a) $f_c = 900\text{ MHz} = 9\times10^8\text{ Hz}$, $R = 100\text{ m}$
$\Downarrow$
worst $d = 100\text{ m}$

Need to make sure that the terminals at the cell boundary receive the minimum required power.

nondirectional antenna $\Rightarrow G_{T_x}G_{R_x} = 1$

By the Friss Equation,

$$\frac{10\mu W}{P_t} = \frac{P_r}{P_t} = \left(\frac{\sqrt{G_{T_x}G_{R_x}}\,c}{4\pi d f}\right)^2 = \left(\frac{\cancel{3}\times\cancel{10^8}}{4\pi\times100\times\underset{3}{\cancel{9}}\times\cancel{10^8}}\right)^2$$

$$P_t = 10\times10^{-6}\times(12\pi\times100)^2 = 144\pi^2\times10^{-1} \approx 142\text{ W}$$

(b) If the system frequency is changed to $f = 5\text{ GHz} = 5\times10^9\text{ Hz}$,

then we need

$$P_t = \frac{P_r}{\left(\frac{\sqrt{G_{T_x}G_{R_x}}\,c}{4\pi d f}\right)^2} = \frac{10\times\cancel{10^{-6}}}{\left(\frac{3\times\cancel{10^8}}{4\pi\times\cancel{100}\times5\times\cancel{10^9}}\right)^2} = \frac{10}{9}\times(4\pi\times5)^2 \approx 4.39\text{ kW}$$

Alternatively, we can first find the power loss when the frequency is changed from 900 MHz to 5 GHz. Under the same P_t, the received power has a loss factor of

$$\left(\frac{0.9}{5}\right)^2.$$

Therefore, to get the same amount of received power, the transmit power must increase by a factor of $\left(\frac{5}{0.9}\right)^2$.

So, we need $P_t = \overset{8}{\cancel{144}}\,\frac{\pi^2}{\cancel{10}} \times \frac{\overset{5}{\cancel{25}}}{\underset{9}{\cancel{81}}} \times 100 = \frac{4000}{9}\pi^2 \approx 4.39\text{ kW}$

Q4

Tuesday, February 14, 2012
9:02 AM

(a) Simplified path loss model:

$$\frac{P_r}{P_t} = K\left(\frac{d_0}{d}\right)^{\gamma}$$

In dB, this is

$$P_r[dB] - P_t[dB] = \overbrace{10\log_{10} K}^{b} + \gamma\, \overbrace{10\log_{10}\left(\frac{d_0}{d}\right)}^{\alpha} \quad \bigstar$$

For free-space path gain, $K = \left(\frac{\lambda}{4\pi d_0}\right)^2 = \left(\frac{c}{4\pi d_0 f}\right)^2$

Here, $f = 900$ MHz, $d_0 = 1$ m

Therefore,

$$\underbrace{10\log_{10} K}_{b} = 10\log_{10}\left(\frac{c}{4\pi d_0 f}\right)^2 \approx -31.53 \text{ dB}$$

Note that $\bigstar$ is of the form

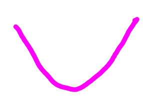
$$y(\alpha) = b + \gamma\alpha$$

We are given five pairs of y_i, α_i.

Want to find γ such that

$$MSE = \sum_{i=1}^{5}\left(y(\alpha_i) - y_i\right)^2 = \underbrace{\sum_i \left(b + \gamma\alpha_i - y_i\right)^2}_{\text{parabola on } \gamma}$$

is minimized.



so, we find

$$\frac{d}{d\gamma} MSE = \sum_i 2\left(b + \gamma\alpha_i - y_i\right)\alpha_i$$

$$\downarrow$$

$$0 = b\sum_i \alpha_i + \gamma\sum_i \alpha_i^2 - \sum_i \alpha_i y_i$$

So, $\gamma = \dfrac{\sum_i \alpha_i y_i - b \sum_i \alpha_i}{\sum_i \alpha_i^2} \approx 3.71$

y_i	d_i	α_i
-70	10	-10
-75	20	-13
-90	50	-17
-110	100	-20
-125	300	-24.77

(b) At $d = 150$ m, $\alpha = 10 \log_{10}\left(\frac{d_0}{d}\right) \approx -21.76$.

$$\Downarrow$$

$$y = b + \gamma\alpha \approx -112.24$$

So, $P_r[dB] - P_t[dB] = -112.24$

$$P_r[dB] = P_t[dB] - 112.24$$

$$P_r[dBm] = \underbrace{P_t[dBm]}_{0} - 112.24 = -112.24 \text{ dBm.}$$

Remark:

If you haven't played with dB and dBm often, you probably find it strange that my answer above does not have the conversion of the unit of -112.24 to dBm.

This is because it is not power. It is simply a number that represents the factor of gain/attenuation.

To see this, let's try an easy example.
Consider two values of power:

$$P_1 = 100 \text{ W} \quad \text{and} \quad P_2 = 100{,}000 \text{ W}$$

Then,

$$P_1 = 10 \log_{10} 100 \text{ dB} = 20 \text{ dB}$$

$$= 10 \log_{10} \frac{100}{1m} \text{ dBm} = 50 \text{ dBm}$$

Similarly,

$$P_2 = 10 \log_{10} 10^5 \text{ dB} = 50 \text{ dB}$$

$$= 10 \log_{10} \frac{10^5}{1m} \text{ dBm} = 80 \text{ dBm}$$

Nothing strange so far...

Now, note that

$$P_2 = 1000 \times P_1$$

In dB, we have

$$\underbrace{P_2 [dB]}_{50 \text{ dB}} = 10 \log_{10} 1000 + \underbrace{P_1 [dB]}_{20 \text{ dB.}}$$

$$= 30 [dB] + P_1 [dB]$$

The number 1,000 is unitless. It is not a quantity that represents power.

Now, note that in dBm, we have

$$\underbrace{P_2 [dBm]}_{80 \text{ dBm}} = \underbrace{30 [dB]}_{\text{still!!}} + \underbrace{P_1 [dBm]}_{50 \text{ dBm}}$$

To avoid confusion, you may see some references use

[dBW] (or [dB(W)]) and [dBmW] (or [dB(mW)])

for the quantities that really represent power.

In which case, we write

$$P_1 [dBW] = 30 [dB] + P_2 [dBW]$$

and

$$P_1 [dBmW] = 30 [dB] + P_2 [dBmW].$$

$P_1\,[\mathrm{dBmW}] = 30\,[\mathrm{dB}] + P_2\,[\mathrm{dBmW}]$.

Still have no "W" because they do not represent power.

Summary: It's OK to directly add or subtract dB values to a power level in dBm. The final answer will be a power level in dBm.

Q5

Wednesday, January 25, 2012
4:31 PM

Let f_i be the center freq. of the i^{th} band group.

The Friss Equation says

$$\frac{P_r}{P_t} = \left(\frac{\sqrt{G_{Tx} G_{Rx}}\, c}{4\pi}\right)\left(\frac{1}{df}\right)^2 = \frac{K}{d^2 f^2}$$

At f_1, the range (max distance) is $d_1 = 10$ m.

So, the min amount of P_r required for the system to work is $P_r = \frac{K}{(d_1 f_1)^2} P_t$.

Now, at f_i, assuming that P_t is the same, then at distance d, the received power is

$$P_r = \frac{K}{(d f_i)^2} P_t .$$

So, to have P_r of at least $\frac{K}{(d_1 f_1)^2} P_t$, which is the min received power for the system to work, we need

$$\frac{\cancel{K}}{(d f_2)^{\cancel{2}}} \cancel{P_t} \geq \frac{\cancel{K}}{(d_1 f_1)^{\cancel{2}}} \cancel{P_t}$$

$$d \leq \underbrace{d_1 \frac{f_1}{f_i}}$$

so, this is the range d_i. (max distance)

Hence,

$$d_i = \frac{\overset{10}{d_1}\, \overset{3960}{f_1}}{f_i}$$

f_i	d_i
5.544	7.14

f_i

f_i	d_i
5,544	7.14
7,128	5.56
8,712	4.55
10,032	3.95

Note that in [Nan, Guo, Qiu, Mo, and Takahashi, 2007], the d_i's are incorrectly calculated by $d_i = d_1 \left(\frac{f_1}{f_i}\right)^2$ which gives 5.10, 3.09, 2.07, and 1.56 respectively.

Q6

Monday, November 12, 2012
2:33 PM

(a) Let $X = k\ln R$. Then, $X \sim \mathcal{N}(\mu, \sigma^2)$ which implies

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

$$\underbrace{R = e^{X/k}}$$

This means $R > 0$.

So, $f_R(r) = 0$ for $r \leq 0$

$$\Rightarrow F_R(r) = P[R \leq r] = P\left[e^{X/k} \leq r\right] \overset{\text{for } r>0}{=} P[X \leq k\ln r]$$

$$= F_X(k\ln r)$$

$$f_R(r) = \frac{d}{dr} F_R(r) \overset{\text{for } r>0}{=} \frac{k}{r} f_X(k\ln r)$$

$$= \frac{k}{r}\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{k\ln r-\mu}{\sigma}\right)^2}$$

So,

$$f_R(r) = \begin{cases} \frac{k}{r}\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{k\ln r-\mu}{\sigma}\right)^2}, & r > 0, \\ 0, & r \leq 0. \end{cases}$$

(b) $$\mathbb{E}R = \int_0^\infty r f_R(r) = \int_0^\infty \frac{k}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{k\ln r-\mu}{\sigma}\right)^2} dr$$

Let $x = \frac{k\ln r - \mu}{\sigma}$ $\Rightarrow dx = \frac{k}{\sigma}\frac{1}{r}dr$

$$\Downarrow$$

$$dr = \frac{\sigma}{k} r\, dx = \frac{\sigma}{k} e^{\frac{\sigma x + \mu}{k}} dx$$

Therefore,

$$\mathbb{E}R = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{\sigma}{k}x} e^{\frac{\mu}{k}} e^{-\frac{1}{2}x^2} dx$$

$-\infty$ ← as $r \to 0$ we have $x \to -\infty$

$$= e^{\mu/k} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2 - 2\frac{\sigma}{k}x + \left(\frac{\sigma}{k}\right)^2\right)} e^{\frac{1}{2}\left(\frac{\sigma}{k}\right)^2} dx$$

$$= e^{\frac{\mu}{k} + \frac{1}{2}\left(\frac{\sigma}{k}\right)^2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x - \frac{\sigma}{k}\right)^2}} dx$$

↳ This is the pdf of a Gaussian RV

whose expected value is σ/k
and variance is 1.
Integrating pdf gives 1.

$$= e^{\frac{\mu}{k} + \frac{1}{2}\left(\frac{\sigma}{k}\right)^2}$$

(c) We want to find r at which $F_R(r) = \frac{1}{2}$.

Recall, from part (a), that $F_R(r) = F_X(k \ln r)$.

By the symmetry of $f_X(x)$ around its expected value μ, we know that $F_X(k \ln r) = \frac{1}{2}$ when $k \ln r = \mu$.

Therefore, our sought value of r is $e^{\mu/k}$.

Q7

Thursday, November 15, 2012
8:45 AM

(a) By definition, $\langle e^{j\beta t}\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{j\beta t}\,dt$

For $\beta = 0$, $\langle e^{j\beta t}\rangle = \langle 1\rangle = 1$.

In general, for any constant c,

$$\langle c\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} c\,dt = \lim_{T\to\infty}\frac{1}{2T}c(2T) = \lim_{T\to\infty} c = c.$$

For $\beta \neq 0$, we apply the Euler's formula : $e^{j\theta} = \cos\theta + j\sin\theta$. Hence,

$$\langle e^{j\beta t}\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\cos\beta t + j\sin\beta t)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos\beta t\,dt + j\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin\beta t\,dt$$

$$= \langle\cos\beta t\rangle + j\langle\sin\beta t\rangle$$

← In general, $\langle a x(t) + b y(t)\rangle = a\langle x(t)\rangle + b\langle y(t)\rangle$

$$= 0 + j0 = 0$$

Time average of cosine or sine is 0

(b)

$$g(t) = \sum_k c_k e^{j2\pi f_k t}$$

$$|g(t)|^2 = g(t)\times g^*(t) = \left(\sum_k c_k e^{j2\pi f_k t}\sum_m c_m^* e^{-j2\pi f_m t}\right)$$

$$= \sum_k\sum_m c_k c_m^* e^{j2\pi(f_k - f_m)t}$$

$$\langle |g(t)|^2\rangle = \sum_k\sum_m c_k c_m^* \underbrace{\langle e^{j2\pi(f_k-f_m)t}\rangle}_{\text{This is 0 when } f_k \neq f_m.} = \sum_k c_k c_k^* \times 1 = \sum_k |c_k|^2.$$

Intuitively, we can say here that $\langle \cdot \rangle$ is the time average operation and the average of cosine or sine is 0.

More rigorously, we start with the fact that for any periodic signal $g(t)$ with period T_0,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, dt = \frac{1}{T_0} \int_{T_0} g(t)\, dt.$$

To see this let's assume

Now, we want to show that $\langle \cos \beta t \rangle = \langle \sin \beta t \rangle = 0$.

To do this, note that

$$= \begin{cases} 1, & \beta = 0 \\ 0, & \beta \neq 0 \end{cases}$$

Q8

Tuesday, November 13, 2012
4:10 PM

(a.i) $g(t) = 3\cos(10t + 30°) = \frac{3}{2}\left(e^{j(10t+30°)} + e^{-j(10t+30°)}\right)$

$= \frac{3}{2}e^{j30°}e^{j10t} + \frac{3}{2}e^{-j30°}e^{-j10t}$

$P_g = \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 2 \times \frac{9}{4} = \frac{9}{2} = 4.5$

Remark: In general, for $g(t) = a\cos(\omega_0 t + \theta)$, $P_g = \frac{a^2}{2}$.

(a.ii) $x(t) = g(t)\cos(10t) = \left(\frac{3}{2}e^{j30°}e^{j10t} + \frac{3}{2}e^{-j30°}e^{-j10t}\right)\left(\frac{1}{2}\left(e^{j10t} + e^{-j10t}\right)\right)$

$$= \frac{3}{4}\left(e^{j30°}e^{j20t} + \underbrace{e^{-j30°}e^{0} + e^{j30°}e^{0}}_{\begin{subarray}{c}\cos(30°) - j\sin(30°) \\ \cos(30°) + j\sin(30°) \\ = 2\cos 30° = \sqrt{3}\end{subarray}} + e^{-j30°}e^{-j20t}\right)$$

$P_g = \left(\frac{3}{4}\right)^2\left(1^2 + (\sqrt{3})^2 + 1^2\right) = \frac{9}{16}(1+3+1) = \frac{45}{16} \approx 2.813$

Remark: In general, for $x(t) = a\cos(\omega_0 t + \theta)\cos(\omega_0 t)$,

$P_x = \left(\frac{a}{4}\right)^2\left(2 + (2\cos\theta)^2\right) = \frac{a^2}{8}\left(1 + 2\cos^2\theta\right)$

Another way to get the formula above is to expand

$x(t) = a\cos(\omega_0 t + \theta)\cos(\omega_0 t)$

$= \frac{a}{2}\left(\cos(2\omega_0 t + \theta) + \cos(\theta)\right)$

$P_x = \frac{a^2}{4}\left(\frac{1}{2} + \cos^2\theta\right)$

(a.iii) $y(t) = g(t)\cos(50t) = \frac{3}{2}\left(e^{j30°}e^{j10t} + e^{-j30°}e^{-j10t}\right)\frac{1}{2}\left(e^{j50t} + e^{-j50t}\right)$

$= \frac{3}{4}\left(e^{j30°}e^{j60t} + e^{-j30°}e^{j40t} + e^{j30°}e^{-j40t} + e^{-j30°}e^{-j60t}\right)$

$P_y = \left(\frac{3}{4}\right)^2\left(1^2 + 1^2 + 1^2 + 1^2\right) = \frac{9}{16} \times 4 = \frac{9}{4} \approx 2.25$

Note that $P_y = \frac{1}{2}P_g$ because $G(f-25)$ and $G(f+25)$ do not over lap.

$P_x \neq \frac{1}{2} P_g$ because $G(f-5)$ and $G(f+5)$ overlap at $f = 0$.

(b.i) $g(t) = 3\cos(10t + 30^\circ) + 4\cos(10t + 120^\circ)$

$$= \mathrm{Re}\left\{ (\underbrace{3\angle 30^\circ + 4\angle 120^\circ}) e^{j10t} \right\} = 5\cos(10t + \theta) \quad \text{where } \theta = 90^\circ$$

$\approx 0.5981 + 4.9641j$

$\approx 5 \angle 83.13^\circ$

$P_g = 5^2 \times \frac{1}{2} = \frac{25}{2} = 12.5$

(b.ii) From part (a.ii), we have

$$P_x = \frac{a^2}{8}(1 + 2\cos^2\theta) = \frac{5^2}{8}(1 + 2\cos^2 83.13^\circ) \approx 3.214$$

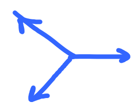
(b.iii) From part (a.iii), we have

$$P_y = \frac{1}{2} P_g = \frac{25}{4} = 6.25$$

(c.i) Look at the three components of $g(t)$ in their phasor representation.

We have $3\angle 0^\circ + 3\angle 120^\circ + 3\angle 240^\circ = 0$

clear when you draw the three vectors



Therefore, $g(t) = 0$. Hence, $P_g = 0$.

(c.ii) $x(t) = 0 \Rightarrow P_x = 0$

(c.iii) $y(t) = 0 \Rightarrow P_y = 0$