1.2 Fourier Transform and Modulation

Wednesday, November 07, 2012
11:27 AM

1.2 Fourier Transform and Modulation

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}\,df \;\underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}}\; X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}\,dt$$

time domain

$$e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f-f_0)$$

$$\delta(t) \xrightarrow{\mathcal{F}} 1$$

$$1 \xrightarrow{\mathcal{F}} \delta(f)$$

$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau = x(t) * y(t) \xrightarrow{\mathcal{F}} X(f)Y(f)$$

$$x(t)y(t) \xrightarrow{\mathcal{F}} X(f) * Y(f) = \int_{-\infty}^{\infty} X(\mu)Y(f-\mu)\,d\mu$$

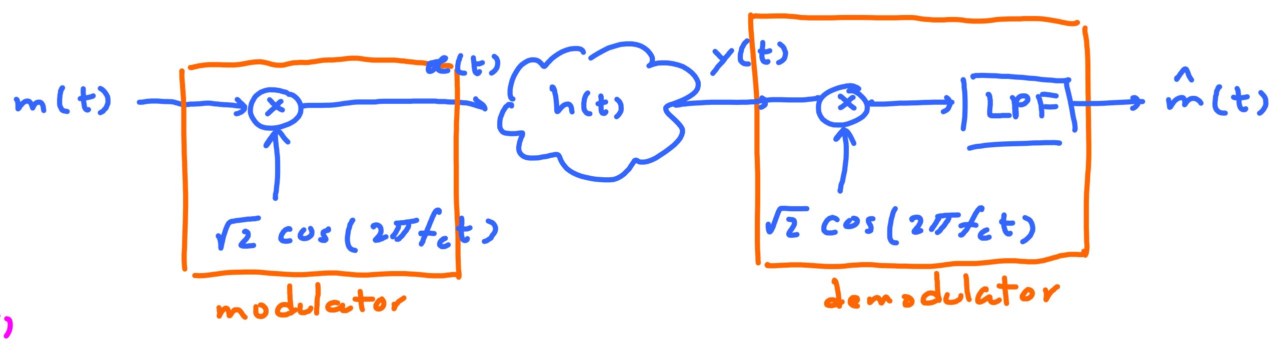
$$x(t)e^{j2\pi f_0 t} \longrightarrow X(f) * \delta(f-f_0) = X(f-f_0)$$

$$x(t)\underbrace{\cos(2\pi f_c t)}_{} \longrightarrow \frac{1}{2}\left(X(f-f_c) + X(f+f_c)\right)$$

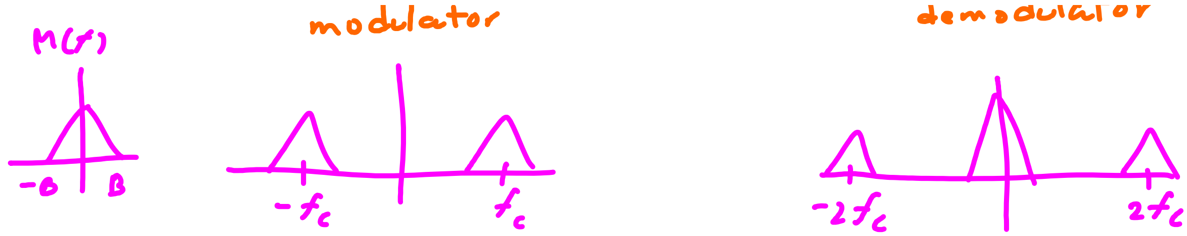
Euler's formula $e^{jx} = \cos(x) + j\sin(x)$

$$\frac{1}{2}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right)$$

Modulation



$M(f)$



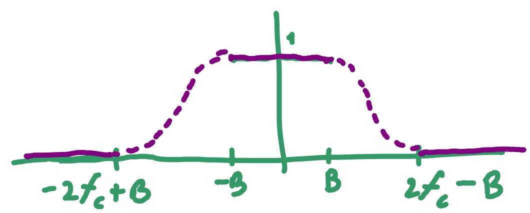
$2f_c - B > B$

$f_c > B$

Assume $m(t)$ is bandlimited ($M(f) = 0$ for $|f| > B$.)

$f_c > B$ (usually, $f_c >> B$)

LPF : $H_{LP}(f)$



1.3 Signal Energy and Power

Wednesday, November 14, 2012
10:34 AM

Consider a signal $g(t)$.

Energy of $g(t)$: $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$

Power of $g(t)$: $P_g = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt$

$= \langle |g(t)|^2 \rangle$

$g(t)$ is an **energy signal** iff $0 < E_g < \infty \Rightarrow P_g = 0$

$g(t)$ is a **power signal** iff $0 < P_g < \infty \Rightarrow E_g = \infty$

If $g(t)$ is periodic with period T_0,

then

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \begin{cases} 0, & g(t) = 0 \text{ a.e.} \\ \infty, & \text{otherwise} \end{cases}$$

So, nonzero periodic signal can't be energy signal.

① $P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2 dt$ (Note that this could be ∞. In which case, $g(t)$ is not a power signal.)

② $g(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_0} t} \Rightarrow P_g = \sum_{k=-\infty}^{\infty} |c_k|^2$

for some coefficients c_k's (Parseval's Theorem)

found by Fourier series expansion.

Examples

① $g(t) = \cos(2\pi f_c t)$

$= \frac{1}{2}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right)$

Euler's formula: $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

directly apply the definition.

$P_g = f_c \int_{1/f_c} |\cos(2\pi f_c t)|^2 dt$ (difficult)

$P_g = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

New operation : $\langle \cdot \rangle \rightarrow$ average in time

New operation : $\langle \cdot \rangle \rightarrow$ average in time

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

$$P_g = \langle |g(t)|^2 \rangle = \langle \cos^2 2\pi f_c t \rangle$$

$$= \langle \tfrac{1}{2}(1+\cos(2\pi\, 2f_c t)) \rangle = \tfrac{1}{2} + 0$$

average of cos = 0.

② $g(t) = a\cos(2\pi f_c t + \phi)$ $\quad P_g = \frac{a^2}{2}$

③ $g(t) = a(t)\cos(2\pi f_c t + \phi)$ $\quad P_g = \frac{P_a}{2}$

Assume $a(t)$ is a power signal
$a(t)$ is bandlimited to $\pm B$
$f_c \gg B$
($A(f \pm f_c)$ do not overlap.)

Motivating example:

$$g(t) = e^{j2\pi t} + e^{j2\pi t} = 2e^{j2\pi t} \Rightarrow P_g = 2^2 = 4$$

Note that we can't say $P_g = 1^2 + 1^2 = 2$ because the two complex exponentials have the same frequency.

④ $g(t) = \sum_k a_k(t)\cos(2\pi f_k t + \phi_k)$ $\quad P_g = \frac{1}{2}\sum_k P_{a_k}$

Assume $A_k(f \pm f_k)$ do not overlap.

⑤ $g(t) = a_1\cos(2\pi f_c t + \phi_1) + a_2\cos(2\pi f_c t + \phi_2)$

(phasor form) $a_1\angle\phi_1 \quad + \quad a_2\angle\phi_2 \quad = a\angle\theta$ ← from calculator

$= a\cos(2\pi f_c t + \theta)$

$$\Big(g(t) = \mathrm{Re}\left\{a_1 e^{j2\pi f_c t} e^{j\phi_1} + a_2 e^{j2\pi f_c t} e^{j\phi_2}\right\}$$

$$= \mathrm{Re}\left\{(a_1 e^{j\phi_1} + a_2 e^{j\phi_2}) e^{j2\pi f_c t}\right\}$$

$$= \mathrm{Re}\left\{a e^{j\theta} e^{j2\pi f_c t}\right\}. \Big)$$

$$P_g = \tfrac{1}{2}a^2$$

If you want an explicit formula ...

$$a_1 e^{j\phi_1} + a_2 e^{j\phi_2} = a_1 e^{j\phi_1}\left(1 + \frac{a_2}{a_1} e^{j(\phi_2-\phi_1)}\right)$$

$$= a_1 e^{j\phi_1}\left(1 + \frac{a_2}{a_1}\cos(\phi_2-\phi_1) + j\frac{a_2}{a_1}\sin(\phi_2-\phi_1)\right)$$

$$= a_1 e^{j\phi_1}\left(1 + \frac{a_2}{a_1}\cos(\phi_2-\phi_1) + j\frac{a_2}{a_1}\sin(\phi_2-\phi_1)\right)$$

$$a^2 = \left|a_1 e^{j\phi_1} + a_2 e^{j\phi_2}\right|^2 = a_1^2\left(\left(1+\frac{a_2}{a_1}\cos(\phi_2-\phi_1)\right)^2 + \left(\frac{a_2}{a_1}\right)^2\sin^2(\phi_2-\phi_1)\right)$$

$$= a_1^2 + 2a_1a_2\cos(\phi_2-\phi_1) + a_2^2$$

Therefore, $P_g = \frac{a_1^2}{2} + \frac{a_2^2}{2} + a_1a_2\cos(\phi_2-\phi_1)$

Alternatively, let $z = a_1e^{j\phi_1} + a_2e^{j\phi_2}$.

without trig. identity

$$a^2 = |z|^2 = zz^* = \left(a_1e^{j\phi_1} + a_2e^{j\phi_2}\right)\left(a_1e^{-j\phi_1} + a_2e^{-j\phi_2}\right)$$

$$= a_1^2 + a_2^2 + a_1a_2e^{-j(\phi-\phi_1)} + a_1a_2e^{j(\phi_2-\phi_1)}$$

$$= a_1^2 + a_2^2 + a_1a_2\,2\cos(\phi_2-\phi_1)$$

1.3 Signal Power Calculation

Monday, November 19, 2012
10:46 AM

Review

$g(t) = e^{j2\pi f_0 t}$ $\quad P_g = \langle |e^{j\cdots}|^2 \rangle = \langle 1^2 \rangle = 1$

$g(t) = \sum_{k=1}^{n} c_k e^{j2\pi f_k t}$ $\quad P_g = \sum_{k=1}^{n} |c_k|^2$ ← You will prove this in HW1

(Assume f_k are distinct.)

Signal Power at Tx

Assume that the power of $m(t)$ is P_m.

$x(t) = m(t) \times \sqrt{2} \cos(2\pi f_c t) \Rightarrow P_x = (\sqrt{2})^2 \frac{1}{2} P_m = P_m = P_t$ ← transmitted power

Assume $M(f \pm f_c)$ do not overlap.

$x(t) \approx \sqrt{P_m} \times \sqrt{2} \cos(2\pi f_c t) \Rightarrow P_x = P_m = P_t$

For simplification, we will assume $m(t)$ is a constant $\sqrt{P_m}$. This assumption avoid having to think about spectral overlapping of the signal.

You can also try to redo the analysis we did in class with $m(t)$ instead of $\sqrt{2P_m}$.

Signal power at Rx

$y(t) = h \times \sqrt{P_m} \times \sqrt{2} \cos\left(2\pi f_c \left(t - \frac{d}{c}\right)\right) = \frac{\alpha}{d} \sqrt{2P_m} \cos\left(2\pi f_c \left(t - \frac{d}{c}\right)\right)$

h: fading coefficient (magnitude)

still unknown here

we will use the Friis equation to derive h.

(free space PL model)

$P_y = h^2 P_t$

$\frac{P_y}{P_t} = h^2 = \left(\frac{\sqrt{G_{Tx} G_{Rx}}\, \lambda}{4\pi d}\right)^2 \Rightarrow h = \underbrace{\frac{\sqrt{G_{Tx} G_{Rx}}\, \lambda}{4\pi}}_{\alpha} \times \frac{1}{d}$

Friis equation

Summary:

The received signal which travel a distance d away from the transmitter is given by

$$y(t) = \frac{\alpha}{d}\sqrt{2P_m}\cos\left(2\pi f_c\left(t - \frac{d}{c}\right)\right).$$

Here, we assume free-space propagation.

Multipath Reception

In general,

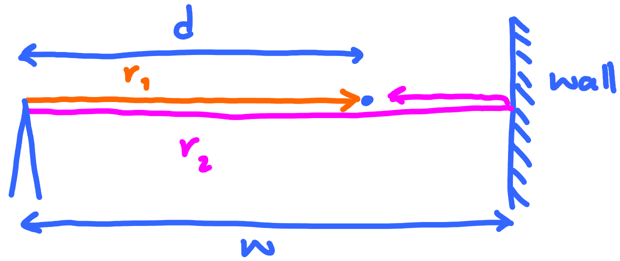
$$y(t) = \sum_{k=1}^{n} R_k \frac{\alpha}{r_k}\sqrt{2P_m}\cos\left(2\pi f_c\left(t - \frac{r_k}{c}\right)\right)$$

propagation distance for the k^{th} path

reflection coefficient

(assume $= 1$ for no reflection
-1 for one reflection.)

Ex. 1.

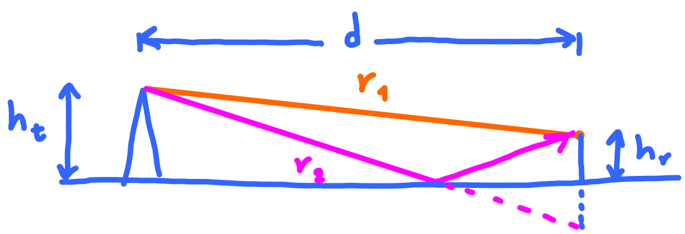


$n = 2$

$r_1 = d$, $R_1 = 1$

$r_2 = 2w - d$, $R_2 = -1$

Ex. 2.



$n = 2$

$r_1 = \sqrt{d^2 + (h_t - h_r)^2}$, $R_1 = 1$

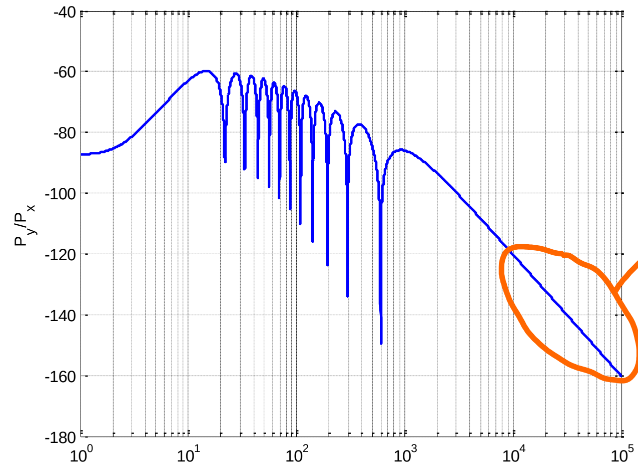
$r_2 = \sqrt{d^2 + (h_t + h_r)^2}$, $R_2 = -1$

Fact: For large d, $r_2 - r_1 \propto \frac{1}{d}$

$$\frac{P_y}{P_x} \propto \frac{1}{d^4}$$

$\gamma = 4$ in the simplified path loss model.

We get $\gamma = 4$ by looking at the plot of $\frac{P_y}{P_x}$



slope ≈ -40

0 1 2 3 4 5

log d

Recall : Simplified path loss model

$$\frac{P_y}{P_x} = \frac{P_r}{P_t} = K\left(\frac{d_0}{d}\right)^{\gamma}$$

In dB, $\quad 10 \log \frac{P_y}{P_x} = 10 \log K d_0^{\gamma} - \underbrace{10\gamma}_{\text{slope}} \log d$

$\Rightarrow \gamma \approx 4$ for the model above.

dB, dBm, dBW

dBmW

dBm, dBW → absolute power → tell signal strength level

dB → relative power → tell the difference (or ratio) between two power levels.

"dB" is used when we compare two power values P_1 and P_2:

$$10 \log_{10} \frac{P_2}{P_1}$$

Rule of Thumb : Double/half power ⇒ add/subtract 3 dB
Ten times/One-tenth power ⇒ add/subtract 10 dB

"dBm" is used when we compare a power with 1 mW :

$$10 \log_{10} \frac{P}{1\,mW}$$

"dBW" is used when we compare a power with 1 W:

$$10 \log_{10} \frac{P}{1\,W}$$

Ex $P = 1\ W = 10 \log \frac{1\,W}{1\,W}$ [dBW] $= 0$ [dBW]

$= 10 \log \frac{1\,W}{1\,mW}$ [dBm] $= 30$ [dBm]

Ex. Now start with $P_1 = 1\ W = 0$ dBW $= 30$ dBm.

Consider $P_2 = 2 \times P_1$. $10 \log_{10} 2 \approx 3$ dB

Then, P_2 is 3 dB more than P_1

$P_2 = 2\,W = 3$ [dBW] $= 33$ [dBm]

Note that we add 3 dB to both 0 dBW and 30 dBm

to get 3 dBW and 33 dBm for the power P_2.

It may look confusing at first that we can directly add 3 dB to quantities in dBW and dBm. It may look less surprising if you remember that dB quantity is simply a (unitless) factor that tells the ratio of the two powers.

3 dB means one power value is twice as much as another.

When you double 1 W, you get 2 W

double 1000 mW, you get 2000 mW.

It does not matter whether you express your power as 1 W or 1000 mW, you simply multiply by 2.

Similarly, in log scale, it does not matter whether you express your power as 0 dBW or 30 dBm, you simply add 3 dB when you double the amount of power.

Sec. 2.4

Old formula : $\text{capacity} = \dfrac{A_{total}}{A_{cell}} \times \dfrac{S}{N}$ usually $\gg S$

\# channels per cell.

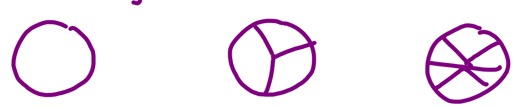
\# cells in the system

New formula : $\dfrac{A_{total}}{A_{cell}} \times n \times \delta$

get this from the Erlang B formula

\# users that a cell can support

\# users that a sector can support

Sectoring	No Sectoring	120°	60°
			
K	6	2	1
δ	1	3	6
$\left\lfloor \frac{S}{N\delta} \right\rfloor = m$	$\left\lfloor \frac{S}{N} \right\rfloor$	$\left\lfloor \frac{S}{N3} \right\rfloor$	$\left\lfloor \frac{S}{N6} \right\rfloor$

$$n = \frac{A}{A_u} = \frac{A}{\lambda_u \times \frac{1}{\mu}}$$

Erlang B formula

$$P_b = \frac{\frac{A^m}{m!}}{\sum_{k=0}^{m} \frac{A^k}{k!}}$$

call blocking probability

$$A = \lambda \times \frac{1}{\mu} \xleftarrow{\times n} A_u = \lambda_u \times \frac{1}{\mu}$$

call request rate

(Poisson process)

average call duration

X = call duration

$X \sim \mathcal{E}(\mu)$

$$f_X(x) = \begin{cases} \mu e^{-\mu x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} \quad , \\ 0, \quad , & \text{otherwise} \end{cases}$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \frac{1}{\mu} \equiv H$$

μ = service "rate"