1.3 Signal Energy and Power

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10:34 AM

Consider a signal $g(t)$.

Energy of $g(t)$: $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$

Power of $g(t)$: $P_g = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt$

$= \langle |g(t)|^2 \rangle$

$g(t)$ is an **energy signal** iff $0 < E_g < \infty \Rightarrow P_g = 0$

$g(t)$ is a **power signal** iff $0 < P_g < \infty \Rightarrow E_g = \infty$

If $g(t)$ is periodic with period T_0,
then

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \begin{cases} 0, & g(t) = 0 \text{ a.e.} \\ \infty, & \text{otherwise} \end{cases}$$

So, nonzero periodic signal can't be energy signal.

① $P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2 dt$ (Note that this could be ∞. In which case, $g(t)$ is not a power signal.)

② $g(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_0} t} \Rightarrow P_g = \sum_{k=-\infty}^{\infty} |c_k|^2$

for some coefficients c_k's (Parseval's Theorem)

found by Fourier series expansion.

Examples

① $g(t) = \cos(2\pi f_c t)$ $\quad P_g = f_c \int_{1/f_c} |\cos(2\pi f_c t)|^2 dt$ (difficult) — directly apply the definition.

$= \frac{1}{2}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right) \longrightarrow P_g = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

Euler's formula: $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

New operation : $\langle \cdot \rangle \rightarrow$ average in time
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New operation : $\langle \cdot \rangle$ → average in time

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

$$P_g = \langle |g(t)|^2 \rangle = \langle \cos^2 2\pi f_c t \rangle$$

$$= \left\langle \frac{1}{2}\left(1+\cos(2\pi\, 2f_c t)\right)\right\rangle = \frac{1}{2} + 0$$

average of cos = 0.

② $g(t) = a\cos(2\pi f_c t + \phi)$ $\qquad P_g = \frac{a^2}{2}$

③ $g(t) = a(t)\cos(2\pi f_c t + \phi)$ $\qquad P_g = \frac{P_a}{2}$

Assume $a(t)$ is a power signal

$a(t)$ is bandlimited to $\pm B$

$f_c \gg B$

($A(f \pm f_c)$ do not overlap.)

Motivating example:

$$g(t) = e^{j2\pi t} + e^{j2\pi t} = 2e^{j2\pi t} \Rightarrow P_g = 2^2 = 4$$

Note that we can't say $P_g = 1^2 + 1^2 = 2$ because the two complex exponentials have the same frequency.

④ $g(t) = \sum_k a_k(t)\cos(2\pi f_k t + \phi_k)$ $\qquad P_g = \frac{1}{2}\sum_k P_{a_k}$

Assume $A_k(f \pm f_k)$ do not overlap.

⑤ $g(t) = a_1\cos(2\pi f_c t + \phi_1) + a_2\cos(2\pi f_c t + \phi_2)$

(Phasor form) $a_1\angle\phi_1 \;+\; a_2\angle\phi_2 \;=\; a\angle\theta$ ← from calculator

$= a\cos(2\pi f_c t + \theta)$

$$\left(g(t) = \mathrm{Re}\left\{a_1 e^{j2\pi f_c t} e^{j\phi_1} + a_2 e^{j2\pi f_c t} e^{j\phi_2}\right\}\right.$$

$$= \mathrm{Re}\left\{\left(a_1 e^{j\phi_1} + a_2 e^{j\phi_2}\right) e^{j2\pi f_c t}\right\}$$

$$\left.= \mathrm{Re}\left\{ a e^{j\theta} e^{j2\pi f_c t}\right\}. \right)$$

$$P_g = \frac{1}{2}a^2$$

If you want an explicit formula ...

$$a_1 e^{j\phi_1} + a_2 e^{j\phi_2} = a_1 e^{j\phi_1}\left(1 + \frac{a_2}{a_1} e^{j(\phi_2-\phi_1)}\right)$$

$$= a_1 e^{j\phi_1}\left(1 + \frac{a_2}{a_1}\cos(\phi_2-\phi_1) + j\frac{a_2}{a_1}\sin(\phi_2-\phi_1)\right)$$

$$= a_1 e^{j\phi_1}\left(1 + \frac{a_2}{a_1}\cos(\phi_2 - \phi_1) + j\frac{a_2}{a_1}\sin(\phi_2 - \phi_1)\right)$$

$$a^2 = \left|a_1 e^{j\phi_1} + a_2 e^{j\phi_2}\right|^2 = a_1^2 \left(\left(1 + \frac{a_2}{a_1}\cos(\phi_2 - \phi_1)\right)^2 + \left(\frac{a_2}{a_1}\right)^2 \sin^2(\phi_2 - \phi_1)\right)$$

$$= a_1^2 + 2a_1a_2\cos(\phi_2 - \phi_1) + a_2^2$$

Therefore, $P_g = \frac{a_1^2}{2} + \frac{a_2^2}{2} + a_1a_2\cos(\phi_2 - \phi_1)$

Alternatively, let $z = a_1 e^{j\phi_1} + a_2 e^{j\phi_2}$.

without trig. identity

$$a^2 = |z|^2 = zz^* = \left(a_1 e^{j\phi_1} + a_2 e^{j\phi_2}\right)\left(a_1 e^{-j\phi_1} + a_2 e^{-j\phi_2}\right)$$

$$= a_1^2 + a_2^2 + a_1a_2 e^{-j(\phi - \phi_1)} + a_1a_2 e^{j(\phi_2 - \phi_1)}$$

$$= a_1^2 + a_2^2 + a_1a_2 2\cos(\phi_2 - \phi_1)$$