



Sirindhorn International Institute of Technology  
Thammasat University at Rangsit  
School of Information, Computer and Communication Technology

## ECS 455: Problem Set 5

**Semester/Year:** 2/2011

**Course Title:** Mobile Communications

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### Due date: Not due

1. Recall that the baseband OFDM modulated signal can be expressed as

$$s(t) = \sum_{k=0}^{N-1} S_k \frac{1}{\sqrt{N}} 1_{[0, T_s]}(t) \exp\left(j \frac{2\pi kt}{T_s}\right)$$

where  $S_0, S_1, \dots, S_{N-1}$  are the (potentially complex-valued) messages.

Let  $T_s = 1$  [ms],  $N = 8$ , and

$$(S_0, S_1, \dots, S_{N-1}) = (1-j, 1+j, 1, 1-j, -1-j, 1, 1-j, -1+j)$$

- a. Use MATLAB `fft` command to plot  $\text{Re}\{s(t)\}$  for  $0 \leq t \leq T_s$ .
- b. Let

i. 
$$a(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \text{Re}\{S_k\} \cos\left(\frac{2\pi kt}{T_s}\right)$$

ii. 
$$b(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \text{Im}\{S_k\} \sin\left(\frac{2\pi kt}{T_s}\right)$$

What is the relationship between  $a(t)$ ,  $b(t)$ , and  $\text{Re}\{s(t)\}$ ?

- c. Let

$$s_2(t) = \sum_{k=0}^{N-1} S_k^* \frac{1}{\sqrt{N}} 1_{[0, T_s]}(t) \exp\left(j \frac{2\pi kt}{T_s}\right).$$

Note the extra conjugation in  $s_2(t)$ .

What is the relationship between  $a(t)$ ,  $b(t)$ , and  $\text{Re}\{s_2(t)\}$ ?

- d. Use MATLAB to plot  $a(t)$  and  $b(t)$  for  $0 \leq t \leq T_s$ .

2. Consider the discrete-time complex FIR channel model

$$y[n] = \{h^* x\}[n] + w[n] = \sum_{m=0}^2 h[m]x[n-m] + w[n]$$

where  $w[n]$  is zero-mean additive Gaussian noise.

In this question, assume that  $h[n]$  has unit energy and that  $H(z)$  has two zeros at

$$z_1 = \rho e^{j\frac{2\pi}{3}} \text{ and } z_2 = \frac{1}{\rho} \text{ where } \rho < 1.$$

a. The information given above implies that

$$h[k] = \frac{1}{\sqrt{E_{h_{im}}}} h_{im}[k] \text{ and } H(z) = \frac{1}{\sqrt{E_{h_{im}}}} H_{im}(z)$$

where

$$h_{im}[k] = \delta[k] - (z_1 + z_2)\delta[k-1] + z_1 z_2 \delta[k-2],$$

$$H_{im}(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) = 1 - (z_1 + z_2)z^{-1} + z_1 z_2 z^{-2},$$

and

$$E_{h_{im}} = 1^2 + |z_1 + z_2|^2 + |z_1|^2 |z_2|^2.$$

**Plot**  $|H(e^{j\omega})| = |H(z)|_{z=e^{j\omega}}$  in the range  $\omega = 0 : \frac{2\pi}{80} : 2\pi$  for  $\rho = 0.5$  and  $0.99$ .

See Appendix A for the discussion on how I get  $h[k]$  and  $H(z)$ .

b. For OFDM system with block size  $N = 8$ , find the corresponding channel gains

$H_k = H(z)|_{z=e^{j\frac{2\pi}{N}k}}$ ,  $k = 0, 1, 2, \dots, N-1$  for  $\rho = 0.5$  and  $0.99$ . In particular, complete the following table.

Ch # $k$	$\rho = 0.5$		$\rho = 0.99$	
	$H_k$	$ H_k $	$H_k$	$ H_k $
0	-0.5455 + 0.1890i	0.5774	-0.0087 + 0.0050i	0.0100
1				
2				
3				
4	0.9820 + 0.5669i	1.1339	0.5860 + 0.9949i	1.1547
5				
6				
7				

3. OFDM simulation: Write a MATLAB code to perform the following operations
- Generate 10,000 OFDM blocks, each is an 8 dimensional QPSK vector. Each element of the vector is independently and randomly chosen from the constellation set  $M = \{1+j, 1-j, -1+j, -1-j\}$ .
  - Perform the MATLAB's IFFT to each vector. Multiply the result by  $\sqrt{N}$ . (See Q1a for the reason why we need to multiply by  $\sqrt{N}$ .)
  - Add the cyclic prefix to each block and transmit over the FIR channel defined in the previous question. Assume  $w[n] \equiv 0$ . Consider two cases:  $\rho = 0.5$  and  $0.99$ .
  - At the receiver, remove the cyclic prefix and perform the FFT to get  $R_k$ . Scale the results by  $1/\sqrt{N}$ .
  - Assume that the receiver knows  $H_k$ . **Detect** the transmitted symbols at each channel. When there is no noise, you have  $R_k = H_k S_k$ . Therefore,  $S_k = \frac{R_k}{H_k}$ . **Record** the symbol error rates (SER) for each channel.
- Remark:** All of them should be 0 in this question. The goal of this problem is to make you that you understand the OFDM system enough to write down MATLAB code for it.

4. (Difficult) Repeat Question 3. However, in this question, the channel noise is generally non-zero.  $w[n]$  is now i.i.d. complex-valued Gaussian noise. Its real and imaginary parts are i.i.d. zero-mean Gaussian with variance  $N_0/2$  where  $N_0 = 1$ .

For the decoder, use the ML (maximum likelihood) detector to decode back  $S_k$ . To do this, first calculate  $\frac{R_k}{H_k}$ . Then, find  $\hat{S}_k$  which is defined to be the closest message  $s$  in the constellation to  $\frac{R_k}{H_k}$ .

- a. Complete the following table. The  $H_k$  can be copied from the Table in Q2b.

	$\rho = 0.5$		$\rho = 0.99$	
Ch # $k$	$ H_k $	SER	$ H_k $	SER
0				
1				
2				
3				
4				
5				
6				
7				

Note that a symbol is counted as decoded correctly if both real and imaginary parts are decoded correctly.

- b. What is the relationship between  $|H_k|$  and SER.

## Appendix A

The channel  $h$  has two zeros at  $z_1 = \rho e^{j\frac{2\pi}{3}}$  and  $z_2 = \frac{1}{\rho}$ .

Thus, before normalized to unit energy, we have

$$H_{un}(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) = 1 - (z_1 + z_2)z^{-1} + z_1 z_2 z^{-2}.$$

Inverse Z-transform gives

$$h_{un}[k] = \delta[k] - (z_1 + z_2)\delta[k-1] + z_1 z_2 \delta[k-2].$$

The energy of  $h_{un}[k]$  is  $E_{h_{un}} = 1^2 + |z_1 + z_2|^2 + |z_1|^2 |z_2|^2$ .

So,

$$h[k] = \frac{1}{\sqrt{E_{h_{un}}}} h_{un}[k] \text{ and } H(z) = \frac{1}{\sqrt{E_{h_{un}}}} H_{un}(z).$$

## Appendix B

When there is some zero-mean Gaussian noise,  $R_k = H_k S_k + W_k$  where  $W_k$  is the noise (in the frequency domain). Therefore,

$$\frac{R_k}{H_k} = S_k + \underbrace{\frac{W_k}{H_k}}_{\text{New Noise}}.$$

Because the noise is Gaussian and zero-mean, the noise will most-likely not take  $\frac{R_k}{H_k}$  too far from  $S_k$ . Therefore, the ML detector gives

$$\hat{S}_k = \arg \min_{s \in M} \{ \|R_k - s H_k\| \} = \arg \min_{s \in M} \left\{ \left\| \frac{R_k}{H_k} - s \right\| \right\},$$

i.e. it detects  $S_k$  as the closest message  $s$  in the constellation to  $\frac{R_k}{H_k}$ . Of course, the noise can

be large and shift  $\frac{R_k}{H_k}$  too far from the original  $S_k$ . Therefore,  $\hat{S}_k$  may be different from  $S_k$ .

This is when symbol error occurs.