HW3 Q1-Q3

Monday, December 21, 2009

1:44 PM

Here are the results from my MATLAB code.

```
p1 = 0.030                                         (1a)
P[N_k = 30] = 0.073                                (1c)
A = 78                                             (1d.ii)
Frequency of occurrence for {N_k = 30} = 0.078     (1d.iii)
Frequency of occurrence for {W_k > 2 mins} = 0.368 (1e)
P[W_k > 2 mins] = 0.368                            (1f)
V = 29946                                          (3a)
D is a geometric r.v. with mean = 33.333 and parameter r =
0.970                                              (3b)
B = 5962                                           (3ci)
The proportion of call requests that were blocked is B/V = 0.199   (3cii)
From Erlang B formula, the blocking probability is 0.200
```

1. $T = 1{,}000$ hrs.
   $\lambda = 30$ arrivals per hour
   $n = 10^6$

(a) p_1 = the probability of exactly one arrival in a slot.

= the mean number of arrivals in a slot (because the random variable is Bernoulli.)

$$= \lambda \times \frac{T}{n} = \frac{30 \times 1000}{10^6} = \boxed{0.03}$$

(b) –

(c) Time is divided into $m = 1000$ non-overlapping intervals.

(i) Mean $= \mathbb{E}N_k = \lambda \times \underbrace{\frac{T}{m}}_{\text{width of each interval}} = 30 \times \frac{1000}{10^3} = \boxed{30}$

(ii), (iii) $P[N_k = 30] = e^{-30}\,\frac{30^{30}}{30!} = \boxed{0.073}$

(Recall that the pmf of a Poisson r.v. is given by

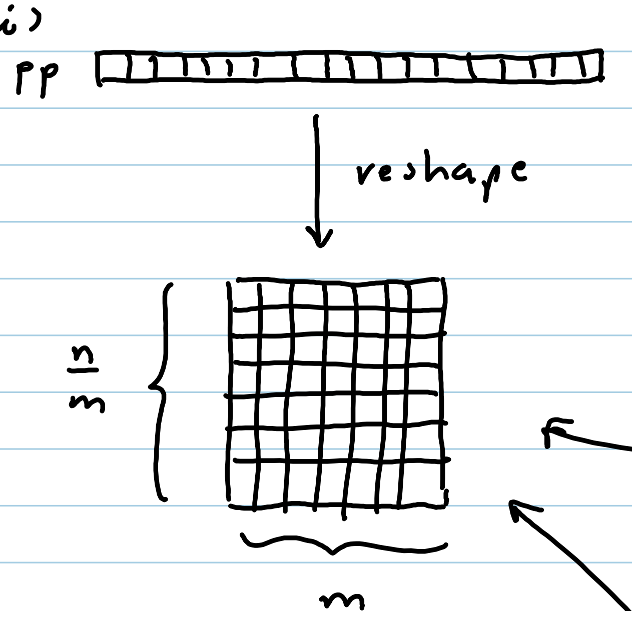
$$p_N(k) = P[N=k] = e^{-\alpha}\,\frac{\alpha^k}{k!}$$

where α is the mean.

(iv) $P[N_k = 10.5] = \boxed{0.}$

Poisson r.v. only takes integer values.

(d) (i)



The command divide the pp (row vector) into m intervals.

Each piece becomes a column of .

$\underbrace{\qquad\qquad}_{m}$

The sum command adds up the 1's in the same column. So, the vector N consists of m member. The k^{th} member is the sum of values in the k^{th} column of . Because the 1's indicate arrivals, the k^{th} member of the vector N is N_k.

(ii) My MATLAB gives $A = 78$.

(iii) $\frac{A}{m} = 0.078$ which is close to the theoretical value of 0.073 in part (c.ii).

(e) 0.368 (MATLAB)
(f) 0.368 (MATLAB) } the same!

② (a) $f_X(x) = \frac{1}{\lambda} e^{-\lambda x}, \quad x > 0$

(b) $\int_a^b f_X(x)\,dx = \int_a^b \frac{1}{\mu} e^{-\mu x}\,dx = -e^{-\mu}\Big|_a^b$

$$= e^{-a\mu} - e^{-b\mu}$$

(c) $a = (k-1)T$ and $b = kT$

$$\int_a^b f_X(a)\,dx = e^{-(k-1)T\mu} - e^{-kT\mu}$$

$$= e^{-(k-1)T\mu}\left(1 - e^{-T\mu}\right)$$

(d) $= e^{-(k-1)T\mu}\left(1 - e^{-T\mu}\right)$, $k = 1, 2, 3, \ldots$

$(1 - e \quad), \quad k = 1, 2, 3, \ldots$

is Geometric with $r = e^{-T\mu}$.

3 (a) $v = 29946$ (MATLAB)

(b) (i) D is geometric : $p[D=k] = (1-r)\,r^{k-1}$

In this case, the T in 2 is replaced by $\frac{T}{n} = \frac{1000}{10^6} = \frac{1}{10^3}$ hr.

$$\text{So,} \quad r = e^{-\frac{T}{n}\mu} \approx 1 - \frac{T}{n}\mu$$

Note that $\frac{1}{\mu} = 2 \text{ min} = \frac{2}{60}$ hr. $= \frac{1}{30}$ hr.

Therefore,

$$r = e^{-\frac{1}{1000} \times 30} \approx 0.97$$

(ii)

For geometric r.v. D with $p_D(k) = (1-r)\,r^{k-1}$

$$\mathbb{E}D = \sum_{k=1}^{\infty} k\,(1-r)\,r^{k-1} = (1-r)\sum_{k=1}^{\infty} k\,r^{k-1}$$

Recall that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Taking $\frac{d}{dr}$ on both sides we have

$$\sum_{k=1}^{\infty} k\,r^{k-1} = -\frac{1}{(1-r)^2}(-1) = \frac{1}{(1-r)^2}$$

$\mathbb{E}D = (1-r) \times \frac{1}{\quad} = \frac{1}{\quad}$

$$\mathbb{E}D = \cancel{(1-r)} \times \frac{1}{(1-r)^{\cancel{2}}} = \frac{1}{1-r}$$

$$\approx \frac{1}{1-\left(1-\frac{I}{n}\mu\right)} = \frac{1}{\frac{I}{n}\mu} = \frac{n}{T} \times \frac{1}{\mu}$$

$$= 10^3 \times \frac{1}{30} \approx \boxed{33.3 \quad \text{slots}}$$

(c) (i) $B = 5962$ (MATLAB)

(ii) $\frac{B}{V} = \boxed{0.199}$ (MATLAB)

Note that $A = \frac{\lambda}{\mu} = 30 \times \frac{1}{30} = 1$

From Erlang B,

$$P_b = \frac{A^2/2!}{1 + A + \frac{A^2}{2}} = \frac{A^2}{2 + 2A + A^2} = \frac{1}{5}$$

$$= \boxed{0.2}$$

almost the same as simulation result.

HW3 Q4

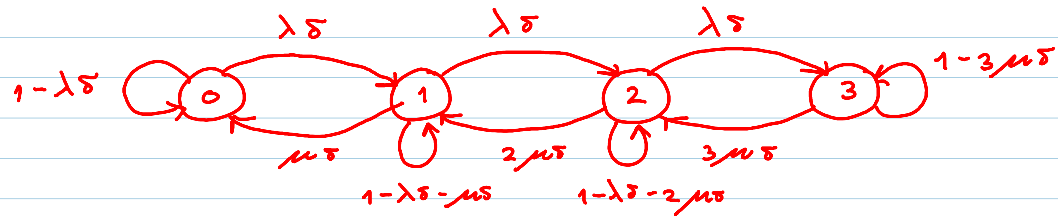
Tuesday, February 14, 2012

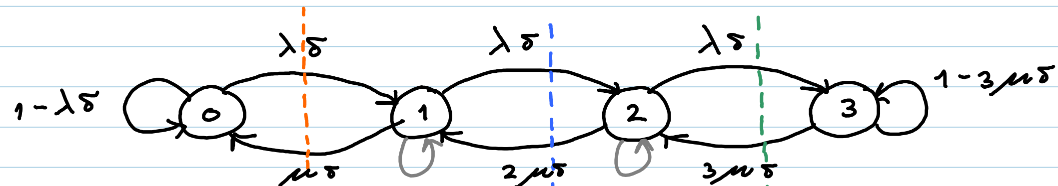
1:04 PM

$m = 3$

(a) Erlang B model

Markov chain:





$p_0 \lambda \cancel{\delta} = p_1 \mu \cancel{\delta}$ $\qquad$ $p_1 \cancel{\lambda\delta} = p_2\, 2\mu \cancel{\delta}$ $\qquad$ $p_2 \cancel{\lambda\delta} = 3\mu\cancel{\delta}\, p_3$

$p_1 = A p_0$ $\qquad$ $p_2 = \frac{A}{2} p_1 = \frac{A^2}{2} p_0$ $\qquad$ $p_3 = \frac{A}{3} p_2 = \frac{A^3}{3!} p_0$

$$p_0 + p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_0 = \left(1 + A + \frac{A^2}{2} + \frac{A^3}{3}\right)^{-1}$$

$$A = \frac{\lambda}{\mu} = \lambda \times \frac{1}{\mu} = \left(\cancel{10}^{\,2} \ \frac{\text{calls}}{\text{hour}} \times \frac{1 \text{ hour}}{\cancel{60}\ \cancel{5} \text{ mins}}\right) \times \left(\cancel{12} \text{ mins}\right) = 2 \text{ Erlangs.}$$

$$\Rightarrow p_0 = \frac{3}{19},\ p_1 = \frac{6}{19},\ p_2 = \frac{6}{19},\ \underbrace{p_3 = \frac{4}{19}}$$

$$\Downarrow$$

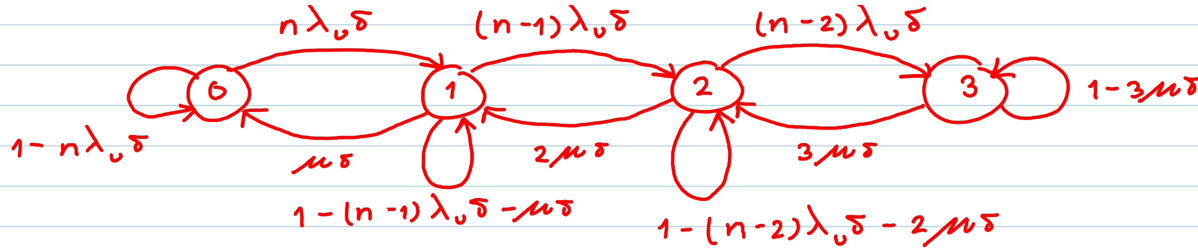
call blocking probability $= \frac{4}{19} \approx 0.211$

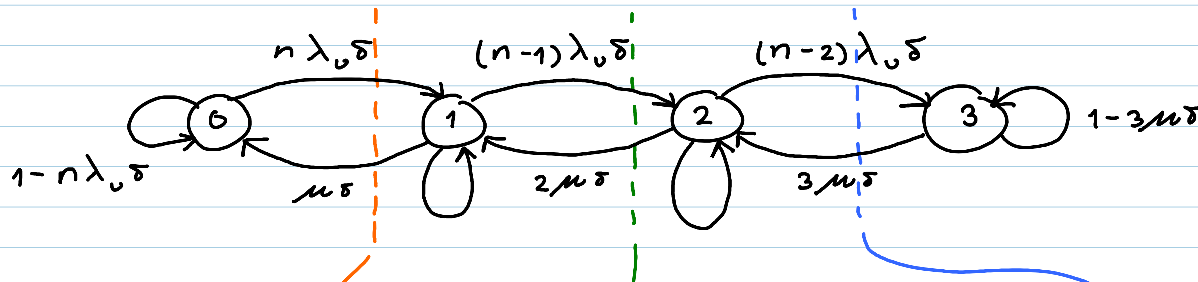
(b) and (c)

Observe that $\lambda_u \times n = \lambda$ in part (a).

so, $\qquad \lambda_u = \frac{\lambda}{n}$ and $A_u = \frac{A}{n} = \frac{2}{n}$ Erlangs.

Markov chain:





$p_0 n\lambda_u \cancel{\delta} = p_1 \mu \cancel{\delta}$ $\qquad$ $p_1(n-1)\lambda_u\cancel{\delta} = 2\mu\cancel{\delta}\, p_2$ $\qquad$ $p_2(n-2)\lambda_u\cancel{\delta} = p_3\, 3\mu\cancel{\delta}$

$p_1 = nA_u$

$$p_2 = \frac{(n-1)}{2}A_u p_1 = \frac{n(n-1)}{2}A_u^2 p_0$$

$$p_3 = \frac{(n-2)}{3}A_u p_2 = \frac{n(n-1)(n-2)A_u^3}{3!}p_0$$

$$p_0+p_1+p_2+p_3 = 1 \Rightarrow \begin{cases} p_0 = \frac{25}{131},\ p_1 = \frac{50}{131},\ p_2 = \frac{40}{131},\ p_3 = \frac{16}{131} & \text{when } n=5, \\ p_0 = 0.159,\ p_1 = 0.319,\ p_2 = 0.316,\ p_3 = 0.206 & \text{when } n=100. \end{cases}$$

As discussed in class, the call blocking probability is given by

$$\frac{(n-m)\,p_m}{\sum_{k=0}^{m}(n-k)\,p_k} = \frac{(n-3)\,p_3}{n\,p_0+(n-1)p_1+(n-2)p_2+(n-3)p_3}$$

$$= \begin{cases} \frac{32}{477} \approx 0.067 & \text{when } n=5 \\ 0.203 & \text{when } n=100 \end{cases}$$

close to the answer from Erlang B.

Remark:

For those who are interested in why the Engset model converges to the Erlang B model when $n\to\infty$, read on.

Note that

$$p_k = \binom{n}{k} A_u^k p_0 = \binom{n}{k} \frac{A^k}{n^k} p_0 = \frac{n!}{(n-k)!\,k!\,n^k} A^k p_0 .$$

For fixed k,

$$\frac{n!}{(n-k)!\,n^k} = \frac{n \times (n-1) \times \cdots \times (n-(k-1))}{n^k} = \frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-(k-1)}{n}$$

$$\to 1 \text{ as } n \to \infty .$$

Hence,

$$p_k = \frac{\binom{n}{k} \frac{A^k}{n^k}}{\sum_{i=0}^{m} \binom{n}{i} \frac{A^i}{n^i}} \to \underbrace{\frac{\frac{1}{k!} A^k}{\sum_{i=0}^{m} \frac{1}{i!} A^i}} \quad \text{as } n \to \infty$$

same as the steady-state probabilities in Erlang B model.

Similarly, for the call blocking probability,

$$P_{CB} = \frac{(n-m) p_m}{\sum_{k=0}^{m} (n-k) p_k} = \frac{(n-m) \binom{n}{m} \frac{A^m}{n^m} \cancel{p_0}}{\sum_{k=0}^{m} (n-k) \binom{n}{k} \frac{A^k}{n^k} \cancel{p_0}} = \frac{\binom{n}{m} \frac{A^m}{n^m}}{\sum_{k=0}^{m} \frac{n-k}{n-m} \binom{n}{k} \frac{A^k}{n^k}}$$

$$\to \underbrace{\frac{\frac{A^m}{m!}}{\sum_{k=0}^{m} \frac{A^k}{k!}}} \quad \text{as } n \to \infty$$

same as the call blocking probability in Erlang B model.

Q5

(a)

$$P_m = \frac{\binom{n}{m} A_u^m}{\sum_{k=0}^{m} \binom{n}{k} A_u^k} = \frac{\sum_{k=0}^{m} \binom{n}{k} A_u^k - \sum_{k=0}^{m-1} \binom{n}{k} A_u^k}{\sum_{k=0}^{m} \binom{n}{k} A_u^k}$$

$$= \frac{z(m,n) - z(m-1,n)}{z(m,n)} = 1 - \frac{z(m-1,n)}{z(m,n)}$$

Hence, $c = 1$

(b)

First, note that

$$(n-k) \times \binom{n}{k} = (n-k) \times \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k-1)!}$$

$$= \frac{n \times (n-1)!}{k!\,(n-1-k)!} = n \binom{n-1}{k}$$

Therefore,

$$P_b = \frac{(n-m)\binom{n}{m} A_u^m}{\sum_{k=0}^{m} (n-k)\binom{n}{k} A_u^k} = \frac{\cancel{n}\binom{n-1}{m} A_u^m}{\sum_{k=0}^{m} \cancel{n}\binom{n-1}{k} A_u^k}$$

$$= \frac{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k - \sum_{k=0}^{m-1} \binom{n-1}{k} A_u^k}{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k}$$

$$= \frac{z(m, n-1) - z(m-1, n-1)}{z(m, n-1)} = 1 - \frac{z(m-1, n-1)}{z(m, n-1)}$$

Hence, $c_1 = c_2 = c_4 = 1$ and $c_3 = 0$.

(c)

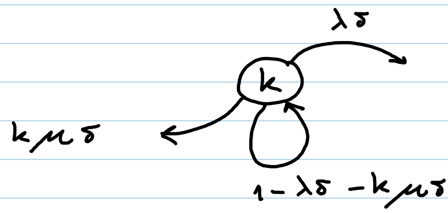
$$P_b = \frac{\binom{n-1}{m} A_u^m}{\sum^{m} \binom{n-1}{k} A^k} \overset{m=n-1}{=} \frac{\binom{m}{m} A_u^m}{\sum^{m} \binom{m}{k} A^k} = \frac{A_u^m}{(1+A)^m} = \left(\frac{A_u}{1+A_u}\right)^m$$

$$P_b = \frac{\binom{n-1}{m}A_u^m}{\sum_{k=0}^{m}\binom{n-1}{k}A_u^k} = \frac{\binom{m}{m}A_u^m}{\sum_{k=0}^{m}\binom{m}{k}A_u^k} = \frac{A_u^m}{(1+A_u)^m} = \left(\frac{A_u}{1+A_u}\right)^m$$

Q6

(a) Nothing changes from the M/M/m/m model when $k < m$.

We have

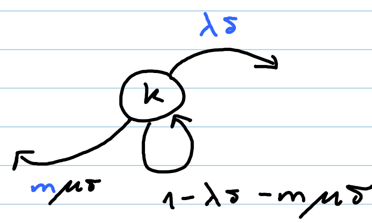


When $k \geq m$, the call request rate is still λ. The difference is that now we have a queue for the new requests to wait. (In M/M/m/m, these requests are discarded and the calls are blocked.)

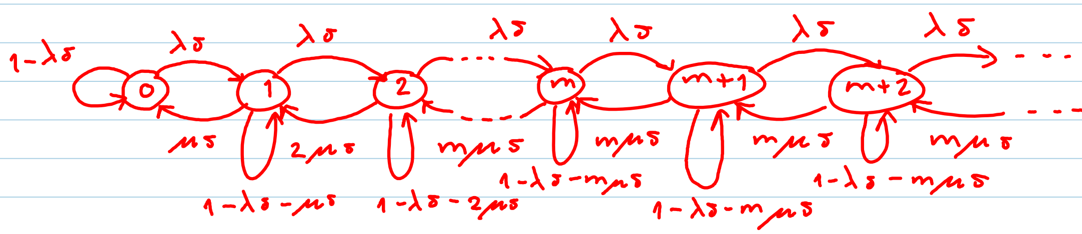
When $k \geq m$, all m channels are being used. There are $k-m$ requests waiting in the queue. When there is one new call request, it will be added to the queue and hence the system move from state k to k+1. Again, this new call request occurs with probability $\lambda\delta$ (approximately).

When $k \geq m$, all m channels are being used. There are m customers talking on the phone. So, the probability of one call ends is (approximately) $m\mu\delta$.

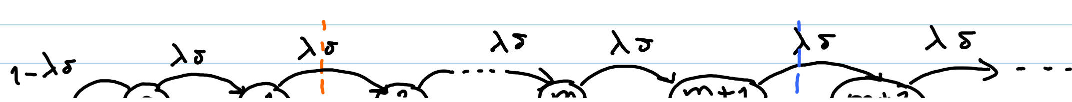
Therefore, when $k \geq m$, we have

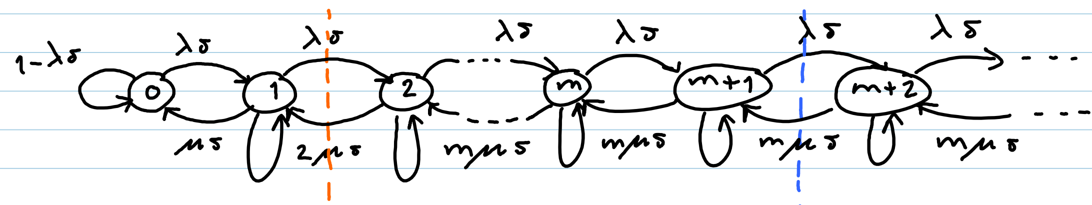


Markov chain:



(b)





$$p_{k-1}\lambda\cancel{\delta} = p_k k\mu\cancel{\delta}$$

$$p_k = \frac{A}{k} p_{k-1}$$

$$= \frac{A^k}{k!} p_0$$

for $0 < k < m$

$$p_{k-1}\lambda\cancel{\delta} = m\mu\cancel{\delta} p_k$$

$$p_k = \frac{A}{m} p_{k-1}$$

for $k \geq m$

$$\Rightarrow p_m = \frac{A}{m} p_{m-1}$$

$$= \frac{A}{m} \frac{A^{m-1}}{(m-1)!} p_0$$

$$= \frac{A^m}{m!} p_0$$

$$p_k = \left(\frac{A}{m}\right)^{k-m} \frac{A^m}{m!} p_0 = \frac{A^k}{m!(m^{k-m})} p_0$$

$$= \frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0 \quad \text{for } k \geq m.$$

$$\sum_{k=0}^{m} p_k = 1 \Rightarrow 1 = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \underbrace{\sum_{k=m}^{\infty} \frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0}_{} = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \frac{A^m}{m!\left(1-\frac{A}{m}\right)} p_0$$

geometric series

$$= \begin{cases} \dfrac{m^m}{m!} \dfrac{\left(\frac{A}{m}\right)^m}{1-\frac{A}{m}} p_0 & \text{if } A < m \\ \infty & \text{if } A \geq m \end{cases}$$

$$\Rightarrow p_0 = \left(\left(\sum_{k=0}^{m-1} \frac{A^k}{k!} \right) + \frac{A^m}{m!\left(1-\frac{A}{m}\right)} \right)^{-1}$$

Therefore,

$$p_k = \begin{cases} \dfrac{A^k}{k!} p_0, & k < m \\ \dfrac{A^k}{m!(m^{k-m})} p_0, & k \geq m \end{cases}$$

(c) Delayed call probability

$(m!(m^{k-m})$

(c) Delayed call probability

$$= \sum_{k=m}^{\infty} P_k = \frac{A^m}{m!\left(1-\frac{A}{m}\right)} P_0 = \frac{\dfrac{A^m}{m!(1-A/m)}}{\dfrac{A^m}{m!(1-\frac{A}{m})} + \sum_{k=0}^{m-1} \dfrac{A^k}{k!}}$$

$$= \frac{A^m}{A^m + m!\left(1-\frac{A}{m}\right)\sum_{k=0}^{m-1}\frac{A^k}{k!}}$$

Remark: This formula is call the "Erlang C formula".

HW3 Q7

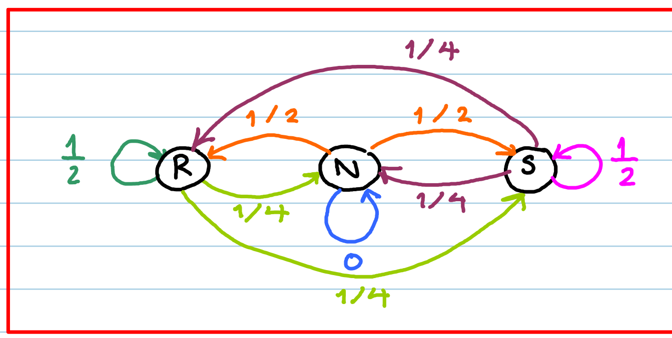
Tuesday, January 11, 2011
8:52 AM

(a)

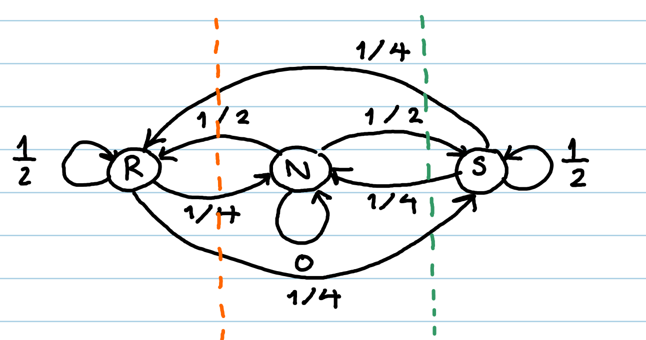
As hinted, we draw three states.

Next, follow the description sentence by sentence to get the transition probabilities.

1. Never have two nice days in a row
2. If have a nice day, just as likely to have snow as rain the next day
3. If have snow or rain, they have an even chance of having the same the next day.
4. If there is change from snow or rain, only half of the time is this a change to a nice day.



(b)



$$\frac{1}{2}p_N+\frac{1}{4}p_S=\frac{1}{4}p_R+\frac{1}{4}p_R \qquad \frac{1}{4}p_S+\frac{1}{4}p_S=\frac{1}{2}p_N+\frac{1}{4}p_R$$

$$\frac{1}{2}p_N+\frac{1}{4}p_S=\frac{1}{2}p_R \qquad \frac{1}{2}p_S=\frac{1}{2}p_N+\frac{1}{4}p_R$$

$$-\frac{1}{2}p_R+\frac{1}{2}p_N+\frac{1}{4}p_S=0 \qquad \frac{1}{4}p_R+\frac{1}{2}p_N-\frac{1}{2}p_S=0$$

One more equation: $p_R+p_N+p_S=1$

solve 3 eqns, 3 unknowns.

One more equation: $p_R + p_N + p_S = 1$

solve 3 eqns, 3 unknowns.

$p_R = 0.4$, $p_N = 0.2$, and $p_S = 0.4$

(c) "365 days" is a long time.

The probability of being a nice day $= p_N = 0.2$

8. Complete the following M/M/m/m description with the following terms:

(I) Bernoulli	(II) binomial	(III) exponential
(IV) Gaussian	(V) geometric	(VI) Poisson

The Erlang B formula is derived under some assumptions. Two important assumptions are (1) the call request process is modeled by a/an ______(A) Poisson________ process and (2) the call durations are assumed to be i.i.d. ______(B) exponential________ random variables. For the call request process, the times between adjacent call requests can be shown to be i.i.d. ______(C) exponential________ random variables. On the other hand, if we consider non-overlapping time intervals, the numbers of call requests in these intervals are ______(D) Poisson________ random variables.

In order to analyze or simulate the system described above, we consider slotted time where the duration of each time slot is small. This technique shifts our focus from continuous-time Markov chain to discrete-time Markov chain. In the limit, for the call request process, only one of the two events can happen during any particular slot: either (1) there is one new call request or (2) there is no new call request. When the slots are small and have equal length, the numbers of new call requests in the slots can be approximated by i.i.d. ______(E) Bernoulli________ random variables. In which case, if we count the total number of call requests during *n* slots, we will get a/an ______(F) binomial________ random variable because it is a sum of i.i.d. ______(E) Bernoulli________ random variables.

When we consider a particular time interval *I* (not necessarily small), the number of slots in this interval will increase as the slots get smaller. In the limit, the number of call requests in the time interval *I* which we approximated by a ______(F) binomial________ random variable before will approach a/an ______(D) Poisson________ random variable.

Similarly, if we consider the numbers of slots between adjacent call requests, these number will be i.i.d. ______(G) Geometric________ random variables. These random variables can be thought of as discrete counterparts of the i.i.d. ______(C) exponential________ random variables in the continuous-time model.

Some term(s) above is/are used more than once. Some term(s) is/are not used.