Sirindhorn International Institute of Technology

Thammasat University at Rangsit

School of Information, Computer and Communication Technology

ECS 455: Problem Set 3

Semester/Year: **2/2011**
Course Title: Mobile Communications
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Due date: Not Due

1. (**MATLAB Simulation**) In this question, we will explore the Poisson process using discrete time approximation as discussed in class. We consider a Poisson process from time 0 to time *T* = 1000. The time unit is irrelevant in this simulation. However, to make the question explicit, we will take the time unit to be in hour. The arrival rate (or the call request rate) is $\lambda = 30$ arrivals per hour. By using the discrete time approximation, we will divide the time interval into n = 1,000,000 slots.
    a. As shown in class, the number of arrivals in the slots can be approximated by i.i.d. Bernoulli random variables with probability p_1 of having exactly one arrival. Find this p_1.
    b. Generate all *n* Bernoulli random variables simultaneously using the command:
    ```
    pp = binornd(1,p1,1,n)
    ```
    c. Let N_k be the total number of arrivals during the interval $\left[\frac{k-1}{m}T, \frac{k}{m}T\right)$ where *k* = 1, 2, …, *m*. In this case, let *m* = 1,000. As shown in class, the random variables $N_1, N_2, \ldots, N_m$ are i.i.d. Poisson random variables.
        i. What is the mean (expected value) of these Poisson random variables?
        ii. Use MATLAB's `poisspdf` function to calculate the probability that $N_1 = 30$.
        iii. Use MATLAB's `poisspdf` function to calculate the probability that $N_{30} = 30$.
        iv. What is the probability that $N_{50} = 10.5$?

d. We can approximate $P[N_1 = 30]$ by calculating the frequency of occurrence from the i.i.d. $N_1, N_2, \ldots, N_m$.

   i. Explain how

```
N = sum(reshape(pp,n/m,m))
```

   gives $N_1, N_2, \ldots, N_m$.

   ii. Let A be the number of $N_1, N_2, \ldots, N_m$ that take the value 30. Find *A*.
   Remark: The variable A in this question is not the same as the variable A in the Erlang B formula.

   iii. The frequency of occurrence is $\frac{A}{m}$. Compare your $\frac{A}{m}$ with the answer from part c.ii and c.iii.

e. The numbers of slots between the adjacent arrivals (1's) in the discrete-time approximation model are given by

`diff(find(pp==1))` [slots].

The corresponding continuous time durations between adjacent arrivals are given by

`W = diff(find(pp==1))*T/n` [hrs].

These time durations are called inter-arrival times. *Use frequency of occurrence* to approximate the probability that the inter-arrival time will be greater than 2 minutes.

f. As mentioned in class, the inter-arrival times $W_1, W_2, W_3, \ldots$ are i.i.d. exponential random variables. Use MATLAB's `expcdf` function to calculate $P[W_{10} > 2 \text{ min}]$. Compare your answer with the answer in the previous part.

2. In this question, we will explore the relationship between exponential random variable and geometric random variable.

   a. Start with an exponential random variable *X* whose mean is $\frac{1}{\mu}$. What is its pdf?

   b. What is the probability that X is in the interval $[a,b)$?

   c. What is the probability that X is in the interval $[(k-1)T, kT)$? Assume *T* is a positive real number and *k* is a positive integer. We will denote this probability by $P[X \in [(k-1)T, kT)]$.

   d. Consider the sequence of number $p_1, p_2, p_3, \ldots$ where $p_k = P[X \in [(k-1)T, kT)]$. Are these p_k's agrees with a pmf of a geometric random variable? Note that the pmf of a geometric random variable can be expressed as $(1-r)r^{k-1}$. Can you find *r* such that the $p_1, p_2, p_3, \ldots$ above satisfies the formula $(1-r)r^{k-1}$?

3. (**MATLAB Simulation**) In this question, we will explore the Erlang B formula using the discrete time approximation. We will use the same Poisson process from the first question (Q1).
   a. Find the total number of arrivals during the time 0 to time *T* = 1000. Denote this number by *V*. What is the your value of *V*?
      This means there will be *V* call requests during our time interval of interest.
   b. For each call request in the previous part, we want to find the call durations. These durations are assumed to be i.i.d. exponential random variables with mean $\frac{1}{\mu}$ = 2 min. So, we can generate all of the call durations by

      ```
      exprnd(2/60,1,V)
      ```

      We want to convert these numbers into numbers of slots. So, we find

      ```
      D = ceil(exprnd(2/60,1,V)/(T/n));
      ```

      i. Express the pmf of D. Approximate the parameter of this pmf.
         Hint: Use question 2 and the fact that $e^{-x} \approx 1-x$ when *x* is small.
      ii. What is the expected value of D?
   c. (Difficult) Suppose there are a total of *m* = 2 channels for this system.
      i. Out of the V call requests, count the number *B* of blocked calls.
      ii. Compare the number $\frac{B}{V}$ with the blocking probability from the Erlang B formula.

4. Consider a system which has <u>3 channels</u>. We would like to find the blocking probability via the Markov chain method. For each of the following models, **<u>draw the Markov chain</u>** via discrete time approximation. Don't forget to indicate the transition probabilities on the arrows. Assume that the duration of each time slot is 1 millisecond. Then, find (1) the **<u>steady-state probabilities</u>** and (2) the long-term **<u>call blocking probability</u>**.
   a. ***Erlang B*** model: Assume that the *<u>total</u>* call request rate is 10 calls per hour and the average call duration is 12 mins.
   b. ***Engset*** model: Assume that there are 5 users. The call request rate for *<u>each</u>* user is 2 calls per hour and the average call duration is 12 mins.

c. ***Engset*** model: Assume that there are 100 users. The call request rate for *each* user is 0.1 calls per hour and the average call duration is 12 mins.

5. (Difficult) In class we have seen that the steady-state probabilities for the Engset model are given by

$$p_i = \frac{\binom{n}{i} A_u^i}{\sum_{k=0}^{m} \binom{n}{k} A_u^k} = \frac{\binom{n}{i} A_u^i}{z(m,n)}, \; 0 \le i \le m,$$

where $z(m,n) = \sum_{k=0}^{m} \binom{n}{k} A^k$.

a. Express p_m (time congestion) in the form $p_m = 1 - \frac{z(m-c,n)}{z(m,n)}$.

What is the value of c?
Hint: c is an integer.

b. The blocked call probability is given by $P_b = \frac{(n-m)\binom{n}{m} A_u^m}{\sum_{k=0}^{m} \binom{n}{k} (n-k) A_u^k}$ which can be rewritten

in the form $P_b = 1 - \frac{z(m-c_1, n-c_2)}{z(m-c_3, n-c_4)}$.

Find c_1, c_2, c_3, c_4.
Hint: c_1, c_2, c_3, c_4 are all integrers.

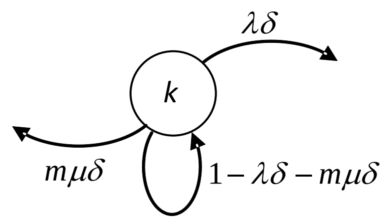
c. Suppose $m = n-1$. Simplify the expression for P_b.

Hint: Your answer should be of the form $(g(A))^m$ for some function *g* of *A*.

6. Consider another modification of the M/M/m/m (Erlang B) system. (There are infinite users) Assume that there is a queue that can be used to hold all requested call which cannot be immediately assigned a channel. This is referred to as an M/M/m/∞ or simply M/M/m system. We will define state *k* as the state where there are k calls in the system. If *k* ≤ *m*, then all of these calls are ongoing. If *k* > *m*, then *m* of them are ongoing and *k-m* of them are waiting in the queue.
Assume that the total call request process is Poisson with rate λ and that the call durations are i.i.d. exponential random variables with expected value 1/μ.

Also assume that $\frac{\lambda}{\mu} < m$.

a. **<u>Draw the Markov chain</u>** via discrete time approximation. Don't forget to indicate the transition probabilities on the arrows.
   Hint: there are infinite number of states. The transition probabilities for state *k* which is < *m* are the same as in the M/M/m/m system. For $k \geq m$, the transition probabilities are given below:



   Explain why the above transition probabilities make sense.

b. Find the **<u>steady-state probabilities</u>**

c. Find the long-term **<u>delayed call probability</u>** (the probability that a call request occurs when all *m* channels are busy and thus has to wait).
   Hint: This will be a summation of many steady-state probabilities. When you simplify your answer, the final answer should be

$$\frac{A^m}{A^m + m!\left(1-\frac{A}{m}\right)\sum_{k=0}^{m-1}\frac{A^k}{k!}}.$$

7. (**Markov Chain**) The Land of Oz is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance of having the same the next day. If there is change from snow or rain, only half of the time is this a change to a nice day.

   a. Draw the Markov chain corresponding to how the weather in the land of Oz changes from one day to the next.
      Hint: This Markov chain will have three states: nice (N), snow (S), and rain (R).
   b. Find the steady-state probabilities.
   c. Suppose it is snowing in the land of Oz today. What is the chance that it will be a nice day next year (365 days later)?

8. Complete the following M/M/m/m description with the following terms:

(I) Bernoulli	(II) binomial	(III) exponential
(IV) Gaussian	(V) geometric	(VI) Poisson

The Erlang B formula is derived under some assumptions. Two important assumptions are (1) the call request process is modeled by a/an ______(A)________ process and (2) the call durations are assumed to be i.i.d. ______(B)________ random variables. For the call request process, the times between adjacent call requests can be shown to be i.i.d. ______(C)________ random variables. On the other hand, if we consider non-overlapping time intervals, the numbers of call requests in these intervals are ______(D)________ random variables.

In order to analyze or simulate the system described above, we consider slotted time where the duration of each time slot is small. This technique shifts our focus from continuous-time Markov chain to discrete-time Markov chain. In the limit, for the call request process, only one of the two events can happen during any particular slot: either (1) there is one new call request or (2) there is no new call request. When the slots are small and have equal length, the numbers of new call requests in the slots can be approximated by i.i.d. _____(E)_________ random variables. In which case, if we count the total number of call requests during *n* slots, we will get a/an _____(F)_________ random variable because it is a sum of i.i.d. _____(E)_________ random variables.

When we consider a particular time interval *I* (not necessarily small), the number of slots in this interval will increase as the slots get smaller. In the limit, the number of call requests in the time interval *I* which we approximated by a _____(F)_________ random variable before will approach a/an ______(D)________ random variable.

Similarly, if we consider the numbers of slots between adjacent call requests, these number will be i.i.d. ______(G)________ random variables. These random variables can be thought of as discrete counterparts of the i.i.d. ______(C)________ random variables in the continuous-time model.

Some term(s) above is/are used more than once. Some term(s) is/are not used.