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ECS 455: Mobile Communications
Fourier Transform and Communication Systems

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Communication systems are usually viewed and analyzed in frequency domain. This note reviews some basic properties of Fourier transform and introduce basic communication systems.

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1 Introduction to communication systems

1.1. Shannon's insight [5]:

> The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.

Definition 1.2. Figure 1 [5] shows a commonly used model for a (single-link or point-to-point) communication system. All information transmission systems involve three major subsystems–a transmitter, the channel, and a receiver.

(a) **Information** source: produce a **message**

- Messages may be categorized as **analog** (continuous) or **digital** (discrete).

(b) **Transmitter**: operate on the message to create a **signal** which can be sent through a channel

(c) **Channel**: the medium over which the signal, carrying the information that composes the message, is sent

- All channels have one thing in common: the signal undergoes **degradation** from transmitter to receiver.
    - Although this degradation may occur at any point of the communication system block diagram, it is customarily associated with the channel alone.
    - This degradation often results from noise and other undesired signals or interference but also may include other distortion effects as well, such as fading signal levels, multiple transmission paths, and filtering.

(d) **Receiver**: transform the signal back into the message intended for delivery

(e) **Destination**: a person or a machine, for whom or which the message is intended

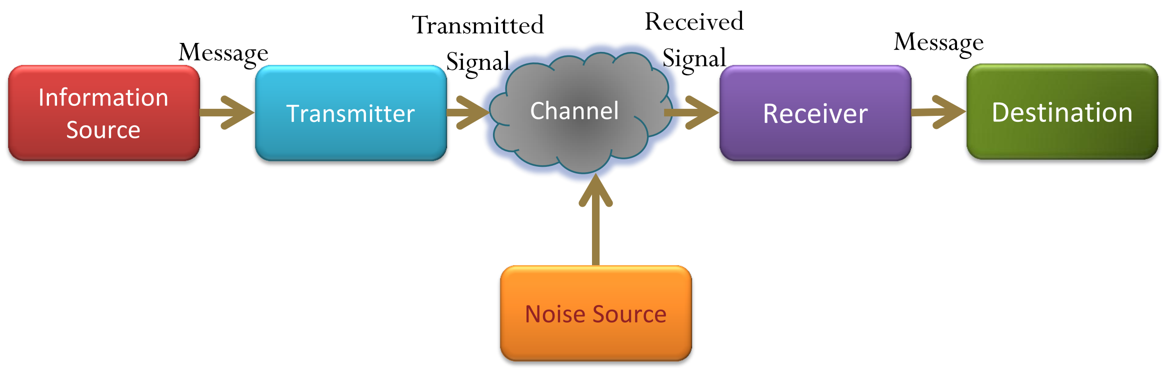


Figure 1: Schematic diagram of a general communication system

2 Frequency-Domain Analysis

Electrical engineers live in the two worlds, so to speak, of time and frequency. Frequency-domain analysis is an extremely valuable tool to the communications engineer, more so perhaps than to other systems analysts. Since the communications engineer is concerned primarily with signal bandwidths and signal locations in the frequency domain, rather than with transient analysis, the essentially steady-state approach of the (complex exponential) **Fourier series** and **transforms** is used rather than the Laplace transform.

2.1 Math background

2.1. ***Euler's formula***: $\boxed{e^{jx} = \cos x + j \sin x.}$

$$\cos(A) = \text{Re}\left\{e^{jA}\right\} = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$
$$\sin(A) = \text{Im}\left\{e^{jA}\right\} = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right).$$

2.2. We can use $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$ and $\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$ to derive many trigonometric identities.

Example 2.3. $\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$

2.4. Similar technique gives

(a) $\cos(-x) = \cos(x)$,

(b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$,

(c) $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, and

(d) the **product-to-sum formula**

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right). \tag{1}$$

2.2 Continuous-Time Fourier Transform

Definition 2.5. The (direct) **Fourier transform** of a signal $g(t)$ is defined by

$$G(f) = \int_{-\infty}^{+\infty} g(t)e^{-j2\pi ft}dt \tag{2}$$

This provides the frequency-domain description of $g(t)$. Conversion back to the time domain is achieved via the **inverse (Fourier) transform**:

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft}df \tag{3}$$

- We may combine (2) and (3) into one compact formula:

$$\boxed{\int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft}df = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi ft}dt.} \tag{4}$$

- We may simply write $G = \mathcal{F}\{g\}$ and $g = \mathcal{F}^{-1}\{G\}$.
- Note that $G(0) = \int g(t)dt$ and $g(0) = \int G(f)df$.

2.6. In some references[1], the (direct) Fourier transform of a signal $g(t)$ is defined by

$$G_2(\omega) = \int_{-\infty}^{+\infty} g(t)e^{-j\omega t}dt \tag{5}$$

[1] MATLAB uses this definition.

In which case, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega)\, e^{j\omega t} d\omega = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_2(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t} dt \tag{6}$$

- In `MATLAB`, these calculations are carried out via the commands `fourier` and `ifourier`.
- Note that $\hat{G}(0) = \int g(t)dt$ and $g(0) = \frac{1}{2\pi}\int G(\omega)d\omega$.
- The relationship between $G(f)$ in (2) and $G_2(\omega)$ in (5) is given by

$$G(f) = G_2(\omega)|_{\omega=2\pi f} \tag{7}$$

$$G_2(\omega) = G(f)|_{f=\frac{\omega}{2\pi}} \tag{8}$$

2.7. Q: The relationship between $G(f)$ in (2) and $G_2(\omega)$ in (5) is given by (7) and (8) which do not involve a factor of 2π in the front. Why then does the factor of $\frac{1}{2\pi}$ shows up in (6)?

Example 2.8. Rectangular and Sinc:

$$1\left[|t| \le a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{\sin(2\pi f a)}{\pi f} = \frac{2\sin(a\omega)}{\omega} = 2a \ \operatorname{sinc}(a\omega) \tag{9}$$

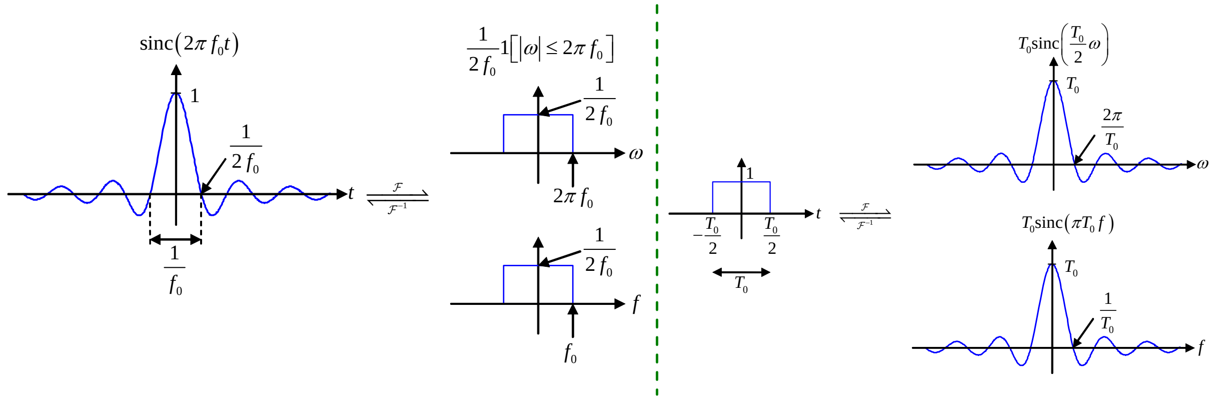


Figure 2: Fourier transform of sinc and rectangular functions

- By setting $a = T_0/2$, we have

$$1\left[|t| \leq \frac{T_0}{2}\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} T_0 \operatorname{sinc}(\pi T_0 f). \tag{10}$$

- In [2, p 78], the function $1[|t| \leq 0.5]$ is defined as the **unit gate** function $\operatorname{rect}(x)$.

Definition 2.9. The function $\operatorname{sinc}(x) \equiv (\sin x)/x$ is plotted in Figure 3.

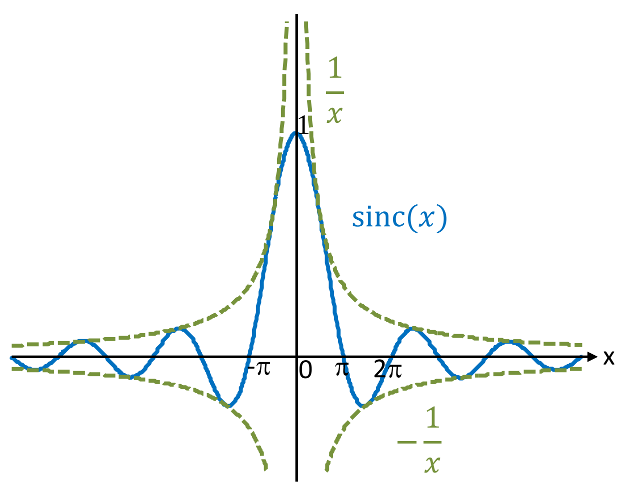


Figure 3: Sinc function

- This function plays an important role in signal processing. It is also known as the filtering or interpolating function.

- Using L'Hôpital's rule, we find $\lim_{x\to 0} \operatorname{sinc}(x) = 1$.
- $\operatorname{sinc}(x)$ is the product of an oscillating signal $\sin(x)$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, $\operatorname{sinc}(x)$ exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.
- In `MATLAB` and in [7, eq. 2.64], $\operatorname{sinc}(x)$ is defined as $(\sin(\pi x))/\pi x$. In which case, it is an even damped oscillatory function with zero crossings at integer values of its argument.

Definition 2.10. The (Dirac) **delta function** or (unit) impulse function is denoted by $\delta(t)$. It is usually depicted as a vertical arrow at the origin. Note that $\delta(t)$ is not a true function; it is undefined at $t = 0$. We define $\delta(t)$ as a generalized function which satisfies the **sampling property** (or **sifting property**)

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \tag{11}$$

for any function $\phi(t)$ which is continuous at $t = 0$. From this definition, It follows that

$$(\delta * \phi)(t) = (\phi * \delta)(t) = \int_{-\infty}^{\infty} \phi(\tau)\delta(t-\tau)d\tau = \phi(t) \tag{12}$$

where we assume that ϕ is continuous at t.

- Intuitively we may visualize $\delta(t)$ as an infinitely tall, infinitely narrow rectangular pulse of unit area: $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} 1\left[|t| \le \frac{\varepsilon}{2}\right]$.

2.11. Properties of $\delta(t)$:

- $\delta(t) = 0$ when $t \neq 0$.
  $\delta(t - T) = 0$ for $t \neq T$.
- $\int_A \delta(t)dt = 1_A(0)$.
  (a) $\int \delta(t)dt = 1$.
  (b) $\int_{\{0\}} \delta(t)dt = 1$.
  (c) $\int_{-\infty}^{x} \delta(t)dt = 1_{[0,\infty)}(x)$. Hence, we may think of $\delta(t)$ as the "derivative" of the unit step function $U(t) = 1_{[0,\infty)}(x)$.

- $\int \phi(t)\delta(t)dt = \phi(0)$ for ϕ continuous at 0.
- $\int \phi(t)\delta(t-T)dt = \phi(T)$ for ϕ continuous at T. In fact, for any $\varepsilon > 0$,

$$\int_{T-\varepsilon}^{T+\varepsilon} \phi(t)\delta(t-T)dt = \phi(T).$$

- $\delta(at) = \frac{1}{|a|}\delta(t)$. In particular,

$$\delta(\omega) = \frac{1}{2\pi}\delta(f) \tag{13}$$

and

$$\delta(\omega - \omega_0) = \delta(2\pi f - 2\pi f_0) = \frac{1}{2\pi}\delta(f - f_0), \tag{14}$$

where $\omega = 2\pi f$ and $\omega_0 = 2\pi f_0$.

Example 2.12. $\delta(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1.$

Example 2.13. $e^{j2\pi f_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \delta\left(f - f_0\right).$

Example 2.14. $e^{j\omega_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi\delta\left(\omega - \omega_0\right).$

Example 2.15. $\cos(2\pi f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(\delta\left(f - f_0\right) + \delta\left(f + f_0\right)\right).$

2.16. Conjugate symmetry[2]: If $x(t)$ is **real**-valued, then $X(-f) = (X(f))^*$

Observe that if we know $X(f)$ for all f positive, we also know $X(f)$ for all f negative. Interpretation: Only half of the spectrum contains all of the information. Positive-frequency part of the spectrum contains all the necessary information. The negative-frequency half of the spectrum can be determined by simply complex conjugating the positive-frequency half of the spectrum.

2.17. ***Shifting*** properties

- ***Time-shift***:

$$g(t - t_1) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{-j2\pi f t_1} G(f)$$

  - Note that $|e^{-j2\pi f t_1}| = 1$. So, the spectrum of $g(t - t_1)$ looks exactly the same as the spectrum of $g(t)$ (unless you also look at their phases).

- ***Frequency-shift*** (or modulation):

$$e^{j2\pi f_1 t} g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G(f - f_1)$$

[2] Hermitian symmetry in [4, p 17].

2.18. Let $g(t)$, $g_1(t)$, and $g_2(t)$ denote signals with $G(f)$, $G_1(f)$, and $G_2(f)$ denoting their respective Fourier transforms.

(a) ***Superposition theorem*** (linearity):

$$a_1 g_1(t) + a_2 g_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} a_1 G_1(f) + a_2 G_2(f).$$

(b) ***Scale-change*** theorem (scaling property [2, p 88]):

$$g(at) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{|a|} G\left(\frac{f}{a}\right).$$

- The function $g(at)$ represents the function $g(t)$ *compress*ed in time by a factor a (when $|a| > l$). Similarly, the function $G(f/a)$ represents the function $G(f)$ *expand*ed in frequency by the same factor a.
- The scaling property says that if we "squeeze" a function in t, its Fourier transform "stretches out" in f. It is not possible to arbitrarily concentrate both a function and its Fourier transform.
- Generally speaking, the more concentrated $g(t)$ is, the more spread out its Fourier transform $G(f)$ must be.
- This trade-off can be formalized in the form of an *uncertainty principle*. See also 2.28 and 2.29.
- Intuitively, we understand that compression in time by a factor a means that the signal is varying more rapidly by the same factor. To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a. Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed.

(c) ***Duality theorem*** (Symmetry Property [2, p 86]):

$$G(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} g(-f).$$

- In words, for any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of a sign change).

  In particular, if the Fourier transform of $g(t)$ is $G(f)$, then the Fourier transform of $G(f)$ with f replaced by t is the original time-domain signal with t replaced by $-f$.
- If we use the ω-definition (5), we get a similar relationship with an extra factor of 2π:

$$G_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi g(-\omega).$$

Example 2.19. $x(t) = \cos(2\pi a f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(\delta(f - af_0) + \delta(f + af_0)\right).$

Example 2.20. From Example 2.8, we know that

$$1\left[|t| \le a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2a \ \operatorname{sinc}\left(2\pi a f\right) \tag{15}$$

By the duality theorem, we have

$$2a \operatorname{sinc}(2\pi a t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1[|-f| \le a],$$

which is the same as

$$\operatorname{sinc}(2\pi f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2f_0} 1[|f| \le f_0]. \tag{16}$$

Both transform pairs are illustrated in Figure 2.

Example 2.21. Let's try to derive the time-shift property from the frequency-shift property. We start with an arbitrary function $g(t)$. Next we will define another function $x(t)$ by setting $X(f)$ to be $g(f)$. Note that f here is just a dummy variable; we can also write $X(t) = g(t)$. Applying the duality theorem to the transform pair $x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X(f)$, we get another transform pair $X(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} x(-f)$. The LHS is $g(t)$; therefore, the RHS must be $G(f)$. This implies $G(f) = x(-f)$. Next, recall the frequency-shift property:

$$e^{j2\pi ct}x\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X\left(f-c\right).$$

The duality theorem then gives

$$X\left(t-c\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{j2\pi c-f}x\left(-f\right).$$

Replacing $X(t)$ by $g(t)$ and $x(-f)$ by $G(f)$, we finally get the time-shift property.

Definition 2.22. The **convolution** of two signals, $x_1(t)$ and $x_2(t)$, is a new function of time, $x(t)$. We write

$$x = x_1 * x_2.$$

It is defined as the integral of the product of the two functions after one is reversed and shifted:

$$x(t) = (x_1 * x_2)(t) \tag{17}$$

$$= \int_{-\infty}^{+\infty} x_1(\mu)x_2(t-\mu)d\mu = \int_{-\infty}^{+\infty} x_1(t-\mu)x_2(\mu)d\mu. \tag{18}$$

- Note that t is a parameter as far as the integration is concerned.
- The integrand is formed from x_1 and x_2 by three operations:
  - (a) time reversal to obtain $x_2(-\mu)$,
  - (b) time shifting to obtain $x_2(-(\mu - t)) = x_2(t-\mu)$, and
  - (c) multiplication of $x_1(\mu)$ and $x_2(t-\mu)$ to form the integrand.
- In some references, (17) is expressed as $x(t) = x_1(t) * x_2(t)$.

Example 2.23. We can get a triangle from convolution of two rectangular waves. In particular,

$$1[|t| \le a] * 1[|t| \le a] = (2a - |t|) \times 1[|t| \le 2a].$$

2.24. Convolution theorem:

(a) Convolution-in-time rule:

$$x_1 * x_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X_1 \times X_2. \tag{19}$$

(b) Convolution-in-frequency rule:

$$x_1 \times x_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X_1 * X_2. \tag{20}$$

Example 2.25. We can use the convolution theorem to "prove" the frequency-sift property in 2.17.

2.26. From the convolution theorem, we have

- $g^2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G * G$
- if g is band-limited to B, then g^2 is band-limited to $2B$

2.27. Parseval's theorem (Rayleigh's energy theorem, Plancherel formula) for Fourier transform:

$$\int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df. \tag{21}$$

The LHS of (21) is called the (total) **energy** of $g(t)$. On the RHS, $|G(f)|^2$ is called the energy spectral density of $g(t)$. By integrating the energy spectral density over all frequency, we obtain the signal 's total energy. The energy contained in the frequency band B can be found from the integral $\int_B |G(f)|^2 df$.

More generally, Fourier transform preserves the inner product [1, Theorem 2.12]:

$$\langle g_1, g_2 \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) df = \langle G_1, G_2 \rangle.$$

2.28. (Heisenberg) **Uncertainty Principle** [1, 6]: Suppose g is a function which satisfies the normalizing condition $\|g\|_2^2 = \int |g(t)|^2 dt = 1$ which automatically implies that $\|G\|_2^2 = \int |G(f)|^2 df = 1$. Then

$$\left(\int t^2 |g(t)|^2 dt \right) \left(\int f^2 |G(f)|^2 df \right) \geq \frac{1}{16\pi^2}, \tag{22}$$

and equality holds if and only if $g(t) = Ae^{-Bt^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$.

- In fact, we have

  $$\left(\int t^2 |g(t - t_0)|^2 dt \right) \left(\int f^2 |G(f - f_0)|^2 df \right) \geq \frac{1}{16\pi^2},$$

  for every t_0, f_0.

- The proof relies on Cauchy-Schwarz inequality.

- For any function h, define its dispersion Δ_h as $\frac{\int t^2 |h(t)|^2 dt}{\int |h(t)|^2 dt}$. Then, we can apply (22) to the function $g(t) = h(t)/\|h\|_2$ and get

  $$\Delta_h \times \Delta_H \geq \frac{1}{16\pi^2}.$$

2.29. A signal cannot be simultaneously time-limited and band-limited.

Proof. Suppose $g(t)$ is simultaneously (1) time-limited to T_0 and (2) band-limited to B. Pick any positive number T_s and positive integer K such that $f_s = \frac{1}{T_s} > 2B$ and $K > \frac{T_0}{T_s}$. The sampled signal $g_{T_s}(t)$ is given by

$$g_{T_s}(t) = \sum_k g[k]\delta\left(t - kT_s\right) = \sum_{k=-K}^{K} g[k]\delta\left(t - kT_s\right)$$

where $g[k] = g\left(kT_s\right)$. Now, because we sample the signal faster than the Nyquist rate, we can reconstruct the signal g by producing $g_{T_s} * h_r$ where the LPF h_r is given by

$$H_r(\omega) = T_s 1[\omega < 2\pi f_c]$$

with the restriction that $B < f_c < \frac{1}{T_s} - B$. In frequency domain, we have

$$G(\omega) = \sum_{k=-K}^{K} g[k]e^{-jk\omega T_s} H_r(\omega).$$

Consider ω inside the interval $I = (2\pi B, 2\pi f_c)$. Then,

$$0 \overset{\omega>2\pi B}{=} G(\omega) \overset{\omega<2\pi f_c}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) e^{-jk\omega T_s} \overset{z=e^{j\omega T_s}}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) z^{-k} \tag{23}$$

Because $z \neq 0$, we can divide (23) by z^{-K} and then the last term becomes a polynomial of the form

$$a_{2K}z^{2K} + a_{2K-1}z^{2K-1} + \cdots + a_1 z + a_0.$$

By fundamental theorem of algebra, this polynomial has only finitely many roots– that is there are only finitely many values of $z = e^{j\omega T_s}$ which satisfies (23). Because there are uncountably many values of ω in the interval I and hence uncountably many values of $z = e^{j\omega T_s}$ which satisfy (23), we have a contradiction. □

3 Modulation and Frequency Shifting

Definition 3.1. The term **baseband** is used to designate the band of frequencies of the signal delivered by the source.

Example 3.2. In telephony, the baseband is the audio band (band of voice signals) of 0 to 3.5 kHz.

Example 3.3. For digital data (sequence of two voltage levels representing 0 and 1) at a rate of R bits per second, the baseband is 0 to R Hz.

Definition 3.4. Modulation is a process that causes a shift in the range of frequencies in a signal.

- The modulation process commonly translates an information-bearing signal to a new spectral location depending upon the intended frequency for transmission.

Definition 3.5. In **baseband communication**, baseband signals are transmitted without modulation, that is, without any shift in the range of frequencies of the signal.

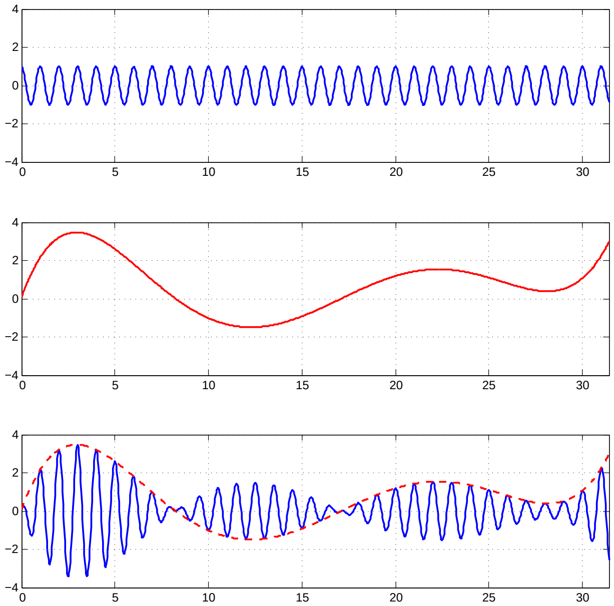
3.6. Recall the frequency-shift property:

$$e^{j2\pi f_c t} g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G\left(f - f_c\right).$$

This property states that multiplication of a signal by a factor $e^{j2\pi f_c t}$ shifts the spectrum of that signal by $f = f_c$.

3.7. Frequency-shifting (frequency translation) in practice is achieved by multiplying $g(t)$ by a sinusoidal:

$$g(t)\cos(2\pi f_c t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(G(f - f_c) + G(f + f_c)\right).$$



Similarly,

$$g(t)\cos(2\pi f_c t + \phi) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(G(f - f_c)e^{j\phi} + G(f + f_c)e^{-j\phi}\right).$$

Definition 3.8. $\cos(2\pi f_c t + \phi)$ is called the (sinusoidal) **carrier signal** and f_c is called the **carrier frequency**. In general, it can also has amplitude A and hence the general expression of the carrier signal is $A\cos(2\pi f_c t + \phi)$.

3.9. Examples of situations where modulation (spectrum shifting) is useful:

(a) **Channel passband matching**: Recall that, for a linear, time-invariant (LTI) system, the input-output relationship is given by

$$y(t) = h(t) * x(t)$$

where $x(t)$ is the input, $y(t)$ is the output, and $h(t)$ is the **impulse response** of the system. In which case,

$$Y(f) = H(f)X(f)$$

where $H(f)$ is called the **transfer function** or **frequency response** of the system. $|H(f)|$ and $\angle H(f)$ are called the **amplitude response** and **phase response**, respectively. Their plots as functions of f show at a glance how the system modifies the amplitudes and phases of various sinusoidal inputs.

(b) **Reasonable antenna size**: For effective radiation of power over a radio link, the antenna size must be on the order of the wavelength of the signal to be radiated.

- Audio signal frequencies are so low (wavelengths are so large) that impracticably large antennas will be required for radiation. Here, shifting the spectrum to a higher frequency (a smaller wavelength) by modulation solves the problem.

(c) **Frequency-division multiplexing** (FDM):

- If several signals, each occupying the same frequency band, are transmitted simultaneously over the same transmission medium, they will all interfere; it will be difficult to separate or retrieve them at a receiver.
- For example, if all radio stations decide to broadcast audio signals simultaneously, the receiver will not be able to separate them.
- One solution is to use modulation whereby each radio station is assigned a distinct carrier frequency. Each station transmits a modulated signal, thus shifting the signal spectrum to its allocated band,

which is not occupied by any other station. A radio receiver can pick up any station by tuning to the band of the desired station.

Definition 3.10. Communication that uses modulation to shift the frequency spectrum of a signal is known as **carrier communication**. [2, p 151]

3.11. A sinusoidal carrier signal $A\cos(2\pi f_c t+\phi)$ has three basic parameters: amplitude, frequency, and phase. Varying these parameters in proportion to the baseband signal results in amplitude modulation (AM), frequency modulation (FM), and phase modulation (PM), respectively. Collectively, these techniques are called **continuous-wave modulation** in [7, p 111].

We will use $m(t)$ to denote the baseband signal. We will assume that $m(t)$ is band-limited to B; that is, $|M(f)| = 0$ for $|f| > B$. Note that we usually call it the message or the modulating signal.

Definition 3.12. The process of recovering the signal from the modulated signal (retranslating the spectrum to its original position) is referred to as **demodulation**, or **detection**.

4 Amplitude modulation: DSB-SC and QAM

Definition 4.1. Amplitude modulation is characterized by the fact that the amplitude A of the carrier $A\cos(2\pi f_c t + \phi)$ is varied in proportion to the baseband (message) signal $m(t)$.

- Because the amplitude is time-varying, we may write the modulated carrier as
$$A(t)\cos(2\pi f_c t + \phi)$$
- Because the amplitude is linearly related to the message signal, this technique is also called **linear modulation**.

4.1 Double-sideband suppressed carrier (DSB-SC) modulation

4.2. Basic idea:

$$\text{LPF}\left\{\underbrace{\left(m\left(t\right)\times\sqrt{2}\cos\left(2\pi f_c t\right)\right)}_{x(t)}\times\left(\sqrt{2}\cos\left(2\pi f_c t\right)\right)\right\}=m\left(t\right). \tag{24}$$

$$\begin{aligned}
x\left(t\right) &= m\left(t\right)\times\sqrt{2}\cos\left(2\pi f_c t\right) = \sqrt{2}m\left(t\right)\cos\left(2\pi f_c t\right)\\
X\left(f\right) &= \sqrt{2}\left(\frac{1}{2}\left(M\left(f-f_c\right)+M\left(f+f_c\right)\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(M\left(f-f_c\right)+M\left(f+f_c\right)\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
v\left(t\right) &= y\left(t\right)\times\sqrt{2}\cos\left(2\pi f_c t\right) = \sqrt{2}x\left(t\right)\cos\left(2\pi f_c t\right)\\
V\left(f\right) &= \frac{1}{\sqrt{2}}\left(X\left(f-f_c\right)+X\left(f+f_c\right)\right)
\end{aligned}$$

Alternatively, we can use the trig. identity from Example 2.3:

$$\begin{aligned} v(t) &= \sqrt{2}x(t)\cos(2\pi f_c t) = \sqrt{2}\left(\sqrt{2}m(t)\cos(2\pi f_c t)\right)\cos(2\pi f_c t) \\ &= 2m(t)\cos^2(2\pi f_c t) = m(t)\left(\cos(2(2\pi f_c t)) + 1\right) \\ &= m(t) + m(t)\cos(2\pi(2f_c)t) \end{aligned}$$

4.3. In the process of modulation, observe that we need $f_c > W$ in order to avoid overlap of the spectra.

4.4. Observe that the modulated signal spectrum centered at f_c, is composed of two parts: a portion that lies above f_c, known as the **upper sideband** (USB), and a portion that lies below f_c, known as the **lower sideband** (LSB). Similarly, the spectrum centered at $-f_c$ has upper and lower sidebands. Hence, this is a modulation scheme with **double sidebands**.

4.2 Quadrature Amplitude Modulation (QAM)

Definition 4.5. One of the possible definition for the ***bandwidth*** (***BW***) of a signal is the difference between the highest frequency and the lowest frequency in the positive-f part of the signal spectrum.

Example 4.6.

4.7. If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz.

This result follows from the application of the width property of convolution[3] to the convolution-in-frequency property. This property states that the width of $x * y$ is the sum of the widths of x and y.

Consequently, if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^n(t)$ is nB Hz.

[3] The width property of convolution does not hold in some pathological cases. See [2, p 98].

4.8. Recall that for real-valued baseband signal $m(t)$, the conjugate symmetry property from 2.16 says that

$$M(-f) = (M(f))^*.$$

The DSB spectrum has two sidebands: the upper sideband (USB) and the lower sideband (LSB), both containing complete information about the baseband signal $m(t)$. As a result, DSB signals occupy twice the bandwidth required for the baseband. To improve the spectral efficiency of amplitude modulation, there exist two basic schemes to either utilize or remove the spectral redundancy:

(a) Single-sideband (SSB) modulation, which removes either the LSB or the USB so that for one message signal $m(t)$, there is only a bandwidth of B Hz.

(b) Quadrature amplitude modulation (QAM), which utilizes spectral redundancy by sending two messages over the same bandwidth of $2B$ Hz.

We will only discussed QAM here. SSB discussion can be found in [7, Section 3.1.3] and [2, Section 4.5].

Definition 4.9. In ***quadrature amplitude modulation*** (***QAM***) or ***quadrature multiplexing***, two baseband signals $m_1(t)$ and $m_2(t)$ are transmitted simultaneously via the corresponding QAM signal:

$$x_{\text{QAM}}(t) = m_1(t)\sqrt{2}\cos(\omega_c t) + m_2(t)\sqrt{2}\sin(\omega_c t).$$

- QAM operates by transmitting two DSB signals via carriers of the same frequency but in phase quadrature.

- QAM can be exactly generated without requiring sharp cutoff bandpass filters.
- Both modulated signals simultaneously occupy the same frequency band.
- The upper channel is also known as the ***in-phase*** (***I***) channel and the lower channel is the ***quadrature*** (***Q***) channel.

4.10. ***Demodulation***: The two baseband signals can be separated at the receiver by synchronous detection:

$$\text{LPF}\left\{x_{\text{QAM}}(t)\sqrt{2}\cos(\omega_c t)\right\} = m_1(t)$$
$$\text{LPF}\left\{x_{\text{QAM}}(t)\sqrt{2}\sin(\omega_c t)\right\} = m_2(t)$$

- $m_1(t)$ and $m_2(t)$ can be separately demodulated.

4.11. Sinusoidal form:

$$x_{\text{QAM}}(t) = \sqrt{2}E(t)\cos(2\pi f_c t + \theta(t)),$$

where

$$E(t) = \sqrt{m_1^2(t) + m_2^2(t)}$$
$$\theta(t) = -\tan^{-1}\left(\frac{m_2(t)}{m_1(t)}\right)$$

4.12. Complex form:

$$x_{\text{QAM}}(t) = \sqrt{2}\text{Re}\left\{(m(t))\,e^{j2\pi f_c t}\right\}$$

where $m(t) = m_1(t) - jm_2(t)$.

- If we use $-\sin(\omega_c t)$ instead of $\sin(\omega_c t)$,

$$x_{\mathrm{QAM}}(t) = m_1(t)\sqrt{2}\cos(\omega_c t) - m_2(t)\sqrt{2}\sin(\omega_c t)$$

  and

$$m(t) = m_1(t) + jm_2(t).$$

- We refer to $m(t)$ as the ***complex envelope*** (or ***complex baseband signal***) and the signals $m_1(t)$ and $m_2(t)$ are known as the ***in-phase*** and ***quadrature(-phase)*** components of $x_{\mathrm{QAM}}(t)$.

- The term "quadrature component" refers to the fact that it is in phase quadrature ($\pi/2$ out of phase) with respect to the in-phase component.

- Key equation:

$$\mathrm{LPF}\left\{\underbrace{\left(\mathrm{Re}\left\{m(t)\times\sqrt{2}e^{j2\pi f_c t}\right\}\right)}_{x(t)}\times\left(\sqrt{2}e^{-j2\pi f_c t}\right)\right\} = m(t).$$

4.13. Three equivalent ways of saying exactly the same thing:

(a) the complex-valued envelope $m(t)$ complex-modulates the complex carrier $e^{j2\pi f_c t}$,

(b) the real-valued amplitude $E(t)$ and phase $\theta(t)$ real-modulate the amplitude and phase of the real carrier $\cos(\omega_c t)$,

(c) the in-phase signal $m_1(t)$ and quadrature signal $m_2(t)$ real-modulate the real in-phase carrier $\cos(\omega_c t)$ and the real quadrature carrier $\sin(\omega_c t)$.

4.14. References: [7, Sect. 2.9.4], [2, Sect. 4.4], and [4, Sect. 1.4.1]

4.15. Question: In engineering and applied science, measured signals are real. Why should real measurable effects be represented by complex signals?

Answer: One complex signal (or channel) can carry information about two real signals (or two real channels), and the algebra and geometry of analyzing these two real signals as if they were one complex signal brings economies and insights that would not otherwise emerge.

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