Q1

(1.a)

As hinted, we draw three states.

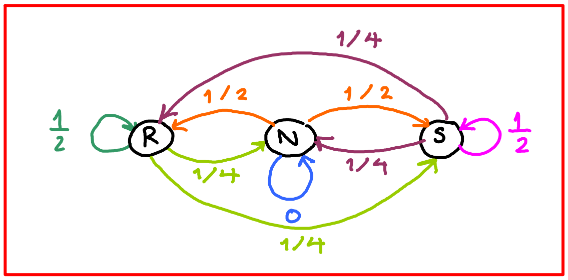
Next, follow the description sentence by sentence to get the transition probabilities.

① Never have two nice days in a row

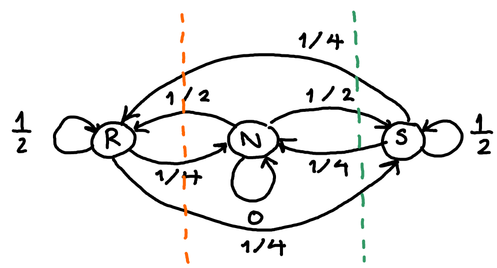
② If have a nice day, just as likely to have snow as rain the next day

③ If have snow or rain, they have an even chance of having the same the next day.

④ If there is change from snow or rain, only half of the time is this a change to a nice day.



(1.b)



$\frac{1}{2}p_N + \frac{1}{4}p_S = \frac{1}{4}p_R + \frac{1}{4}p_R$

$\frac{1}{2}p_N + \frac{1}{4}p_S = \frac{1}{2}p_R$

$-\frac{1}{2}p_R + \frac{1}{2}p_N + \frac{1}{4}p_S = 0$

$\frac{1}{4}p_S + \frac{1}{4}p_S = \frac{1}{2}p_N + \frac{1}{4}p_R$

$\frac{1}{2}p_S = \frac{1}{2}p_N + \frac{1}{4}p_R$

$\frac{1}{4}p_R + \frac{1}{2}p_N - \frac{1}{2}p_S = 0$

One more equation: $p_R + p_N + p_S = 1$

solve 3 eqns, 3 unknowns.

$p_R = 0.4$, $p_N = 0.2$, and $p_S = 0.4$

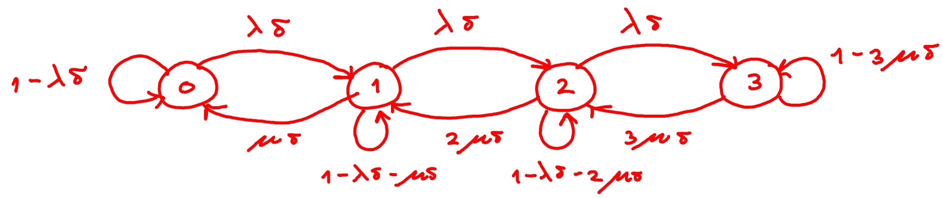
(1.c) "365 days" is a long time.

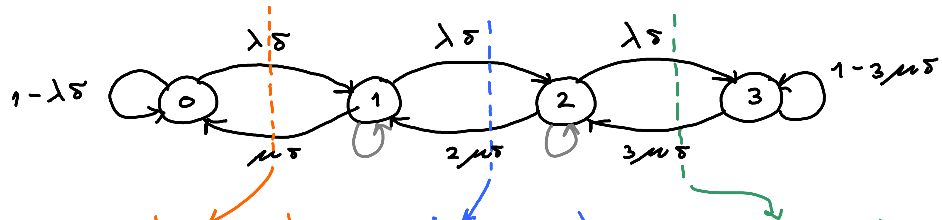
The probability of being a nice day $= p_N = 0.2$

Q2 $m = 3$

(a) Erlang B model

Markov chain:





$p_0 \lambda \cancel{\delta} = p_1 \mu \cancel{\delta}$ $\qquad$ $p_1 \lambda \cancel{\delta} = p_2 \, 2\mu \cancel{\delta}$ $\qquad$ $p_2 \lambda \cancel{\delta} = 3\mu \cancel{\delta} \, p_3$

$p_1 = A p_0$ $\qquad$ $p_2 = \frac{A}{2} p_1 = \frac{A^2}{2} p_0$ $\qquad$ $p_3 = \frac{A}{3} p_2 = \frac{A^3}{3!} p_0$

$$p_0 + p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_0 = \left(1 + A + \frac{A^2}{2} + \frac{A^3}{3}\right)^{-1}$$

$$A = \frac{\lambda}{\mu} = \lambda \times \frac{1}{\mu} = \left(\overset{2}{\cancel{10}} \ \frac{\text{calls}}{\text{hour}} \times \frac{1\ \text{hour}}{\underset{5}{\cancel{60}}\ \text{mins}} \right) \times \left(\cancel{12}\ \text{mins} \right) = 2 \text{ Erlangs.}$$

$$\Rightarrow p_0 = \frac{3}{19}, \quad p_1 = \frac{6}{19}, \quad p_2 = \frac{6}{19}, \quad \underbrace{p_3 = \frac{4}{19}}$$

$$\Downarrow$$

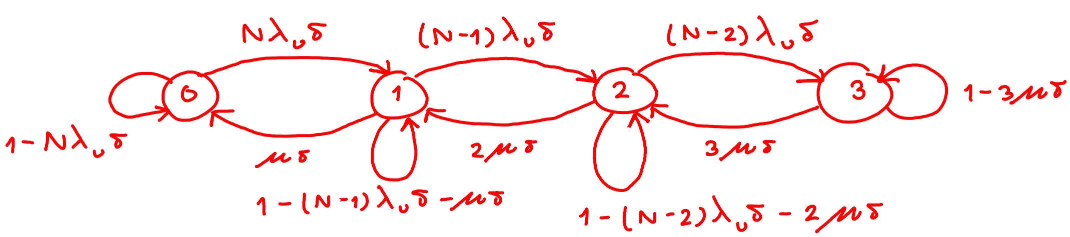
$$\text{call blocking probability} = \frac{4}{19} \approx 0.211$$

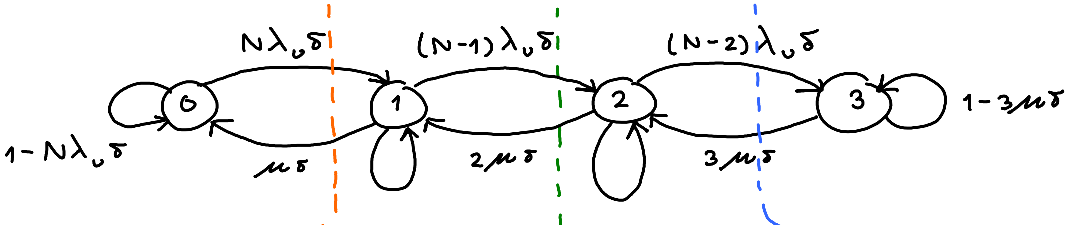
(b) and (c)

Observe that $\lambda_u \times N = \lambda$ in part (a).

So, $\quad \lambda_u = \frac{\lambda}{N}$ and $A_u = \frac{A}{N} = \frac{2}{N}$ Erlangs.

Markov chain:





$p_0 N\lambda_v\cancel{\delta} = p_1\mu\cancel{\delta}$ $\quad$ $p_1(N-1)\lambda_v\cancel{\delta} = 2\mu\cancel{\delta}\, p_2$ $\quad$ $p_2(N-2)\lambda_v\cancel{\delta} = p_3\, 3\mu\cancel{\delta}$

$p_1 = NA_v$

$$p_2 = \frac{(N-1)}{2}A_v p_1 = \frac{N(N-1)}{2}A_v^2 p_0$$

$$p_3 = \frac{(N-2)}{3}A_v p_2 = \frac{N(N-1)(N-2)A_v^3}{3!}p_0$$

$$p_0+p_1+p_2+p_3 = 1 \quad \Rightarrow \begin{cases} p_0 = \frac{25}{131},\ p_1 = \frac{50}{131},\ p_2 = \frac{40}{131},\ p_3 = \frac{16}{131} & \text{when } N=5, \\ p_0 = 0.159,\ p_1 = 0.319,\ p_2 = 0.316,\ p_3 = 0.206 & \text{when } N=100. \end{cases}$$

As discussed in class, the call blocking probability is given by

$$\frac{(N-m)p_m}{\sum_{k=0}^{m}(N-k)p_k} = \frac{(N-3)p_3}{Np_0+(N-1)p_1+(N-2)p_2+(N-3)p_3}$$

$$= \begin{cases} \frac{32}{477} \approx 0.067 & \text{when } N=5 \\ 0.203 & \text{when } N=100 \end{cases}$$

close to the answer from Erlang B.

Remark:

For those who are interested in why the Engset model converges to the Erlang B model when $N\to\infty$, read on.

Note that

$$p_k = \binom{N}{k}A_v^k p_0 = \binom{N}{k}\frac{A^k}{N^k}p_0 = \frac{N!}{(N-k)!\,k!\,N^k}A^k p_0 .$$

For fixed k,

$$\frac{N!}{(N-k)!\,N^k} = \frac{N\times(N-1)\times\cdots\times(N-(k-1))}{N^k} = \frac{N}{N}\times\frac{N-1}{N}\times\cdots\times\frac{N-(k-1)}{N}$$

$$\longrightarrow 1 \quad \text{as} \quad N\to\infty .$$

Hence,

$$p_k = \frac{\binom{N}{k}\frac{A^k}{N^k}}{\sum_{i=0}^{m}\binom{N}{i}\frac{A^i}{N^i}} \longrightarrow \frac{\frac{1}{k!}A^k}{\sum_{i=0}^{m}\frac{1}{i!}A^i} \quad \text{as} \quad N \to \infty$$

same as the steady-state probabilities in Erlang B model.

Similarly, for the call blocking probability,

$$P_{CB} = \frac{(N-m)\,p_m}{\sum_{k=0}^{m}(N-k)\,p_k} = \frac{(N-m)\binom{N}{m}\frac{A^m}{N^m}\cancel{p_0}}{\sum_{k=0}^{m}(N-k)\binom{N}{k}\frac{A^k}{N^k}\cancel{p_0}} = \frac{\binom{N}{m}\frac{A^m}{N^m}}{\sum_{k=0}^{m}\frac{N-k}{N-m}\binom{N}{k}\frac{A^k}{N^k}}$$

$$\longrightarrow \frac{\frac{A^m}{m!}}{\sum_{k=0}^{m}\frac{A^k}{k!}} \quad \text{as} \quad N \to \infty$$

$\hookrightarrow$ same as the call blocking probability in Erlang B model.

HW4 Sol Q3-4

Wednesday, January 05, 2011

3:21 PM

Q3

(3.a)

$$P_m = \frac{\binom{N}{m} A^m}{\sum_{k=0}^{m} \binom{N}{k} A^k} = \frac{\sum_{k=0}^{m} \binom{N}{k} A^k - \sum_{k=0}^{m-1} \binom{N}{k} A^k}{\sum_{k=0}^{m} \binom{N}{k} A^k}$$

$$= \frac{z(m,N) - z(m-1,N)}{z(m,N)} = 1 - \frac{z(m-1,N)}{z(m,N)}$$

Hence, $c = 1$

(3.b)

First, note that

$$(N-k) \times \binom{N}{k} = (N-k) \times \frac{N!}{k!\,(N-k)!} = \frac{N!}{k!\,(N-k-1)!}$$

$$= \frac{N \times (N-1)!}{k!\,(N-1-k)!} = N \binom{N-1}{k}$$

Therefore,

$$P_E = \frac{(N-m) \binom{N}{m} A^m}{\sum_{k=0}^{m} (N-k) \binom{N}{k} A^k} = \frac{\cancel{N} \binom{N-1}{m} A^m}{\sum_{k=0}^{m} \cancel{N} \binom{N-1}{k} A^k}$$

$$= \frac{\sum_{k=0}^{m} \binom{N-1}{k} A^k - \sum_{k=0}^{m-1} \binom{N-1}{k} A^k}{\sum_{k=0}^{m} \binom{N-1}{k} A^k}$$

$$= \frac{z(m, N-1) - z(m-1, N-1)}{z(m, N-1)} = 1 - \frac{z(m-1, N-1)}{z(m, N-1)}$$

Hence, $c_1 = c_2 = c_4 = 1$ and $c_3 = 0$.

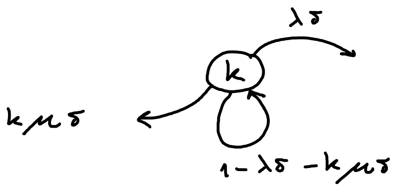
(3.c)

$$P_E = \frac{\binom{N-1}{m} A^m}{\sum_{k=0}^{m} \binom{N-1}{k} A^k} \overset{m=N-1}{=} \frac{\binom{m}{m} A^m}{\sum_{k=0}^{m} \binom{m}{k} A^k} = \frac{A^m}{(1+A)^m} = \left(\frac{A}{1+A}\right)^m$$

Q4

(a) Nothing changes from the M/M/m/m model when $k < m$.

We have

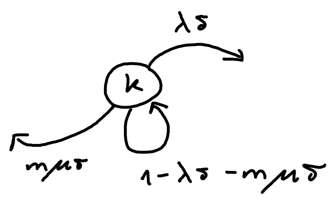


When $k \geq m$, the call request rate is still λ. The difference is that now we have a queue for the new requests to wait. (In M/M/m/m, these requests are discarded and the calls are blocked.)

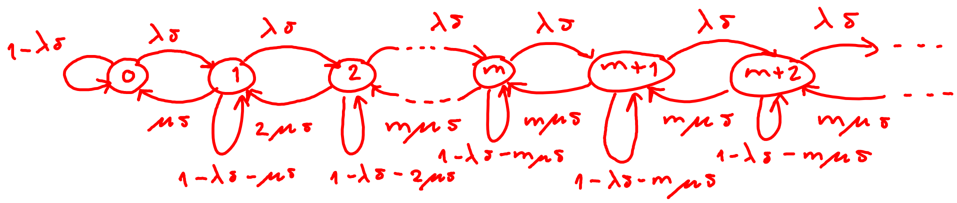
When $k \geq m$, all m channels are being used. There are $k-m$ requests waiting in the queue. When there is one new call request, it will be added to the queue and hence the system move from state k to $k+1$. Again, this new call request occurs with probability $\lambda\delta$ (approximately).

When $k \geq m$, all m channels are being used. There are m customers talking on the phone. So, the probability of one call ends is (approximately) $m\mu\delta$.

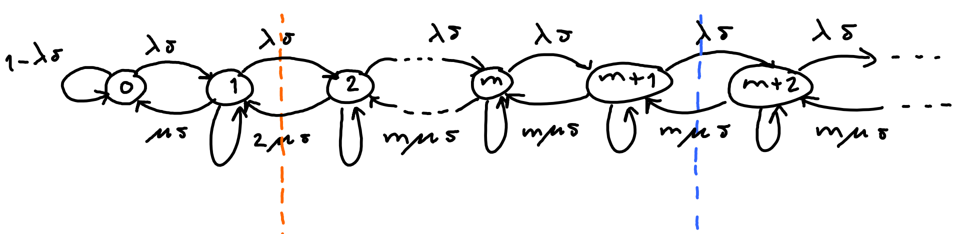
Therefore, when $k \geq m$, we have



Markov chain :



(b)



$p_{k-1}\,\lambda\cancel{\delta} = p_k\, k\mu\cancel{\delta}$

$p_k = \frac{A}{k}\, p_{k-1}$

$= \frac{A^k}{k!}\, p_0$

for $0 < k < m$

$p_{k-1}\,\lambda\cancel{\delta} = m\mu\cancel{\delta}\, p_k$

$p_k = \frac{A}{m}\, p_{k-1}$

for $k \geq m$

$\Rightarrow p_m = \frac{A}{m}\, p_{m-1}$

$m-1$

$$= \frac{A}{m} \frac{A^{\cdots}}{(m-1)!} p_0$$

$$= \frac{A^m}{m!} p_0$$

$$p_k = \left(\frac{A}{m}\right)^{k-m} \frac{A^m}{m!} p_0 = \frac{A^k}{m!(m^{k-m})} p_0$$

$$= \frac{m^m}{m!} \left(\frac{A}{m}\right)^k p_0 \quad \text{for } k \geq m.$$

$$\sum_{k=0}^{m} p_k = 1 \Rightarrow 1 = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \underbrace{\sum_{k=m}^{\infty} \frac{m^m}{m!} \left(\frac{A}{m}\right)^k p_0} = \sum_{k=0}^{m-1} \frac{A^k}{k!} p_0 + \frac{A^m}{m!\left(1-\frac{A}{m}\right)} p_0$$

geometric series

$$= \begin{cases} \frac{m^m}{m!} \frac{\left(\frac{A}{m}\right)^m}{1-\frac{A}{m}} p_0 & \text{if } A < m \\ \infty & \text{if } A \geq m \end{cases}$$

$$\Rightarrow p_0 = \left(\left(\sum_{k=0}^{m-1} \frac{A^k}{k!} \right) + \frac{A^m}{m!\left(1-\frac{A}{m}\right)} \right)^{-1}$$

Therefore,

$$p_k = \begin{cases} \frac{A^k}{k!} p_0, & k < m \\ \frac{A^k}{m!(m^{k-m})} p_0, & k \geq m \end{cases}$$

(c) Delayed call probability

$$= \sum_{k=m}^{\infty} p_k = \frac{A^m}{m!\left(1-\frac{A}{m}\right)} p_0 = \frac{\frac{A^m}{m!(1-A/m)}}{\frac{A^m}{m!\left(1-\frac{A}{m}\right)} + \sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

$$= \frac{A^m}{A^m + m!\left(1-\frac{A}{m}\right) \sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

Remark: This formula is call the "Erlang C formula".