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ECS 455: Mobile Communications
Fourier Transform and Communication Systems

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Communication systems are usually viewed and analyzed in frequency domain. This note reviews some basic properties of Fourier transform and introduce basic communication systems.

Contents

1 Introduction to Signals

1.1 Dirac Delta Function

The (Dirac) delta function or (unit) impulse function is denoted by $\delta(t)$. It is usually depicted as a vertical arrow at the origin. Note that $\delta(t)$ is not a true function; it is undefined at $t = 0$. We define $\delta(t)$ as a generalized function which satisfies the sampling property (or sifting property)

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)$$

for any function $\phi(t)$ which is continuous at $t = 0$. From this definition, It follows that

$$(\delta * \phi)(t) = (\phi * \delta)(t) = \int_{-\infty}^{\infty} \phi(\tau)\delta(t-\tau)d\tau = \phi(t)$$

where we assume that ϕ is continuous at t. Intuitively we may visualize $\delta(t)$ as an infinitely tall, infinitely narrow rectangular pulse of unit area: $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} 1\left[|t| \leq \frac{\varepsilon}{2}\right]$.

We list some interesting properties of $\delta(t)$ here.

- $\delta(t) = 0$ when $t \neq 0$.
  $\delta(t - T) = 0$ for $t \neq T$.
- $\int_A \delta(t)dt = 1_A(0)$.
  (a) $\int \delta(t)dt = 1$.
  (b) $\int_{\{0\}} \delta(t)dt = 1$.
  (c) $\int_{-\infty}^{x} \delta(t)dt = 1_{[0,\infty)}(x)$. Hence, we may think of $\delta(t)$ as the "derivative" of the unit step function $U(t) = 1_{[0,\infty)}(x)$.
- $\int \phi(t)\delta(t)dt = \phi(0)$ for ϕ continuous at 0.
- $\int \phi(t)\delta(t-T)dt = \phi(T)$ for ϕ continuous at T. In fact, for any $\varepsilon > 0$,

$$\int_{T-\varepsilon}^{T+\varepsilon} \phi(t)\delta(t-T)dt = \phi(T).$$

- $\delta(at) = \frac{1}{|a|}\delta(t)$. In particular,

$$\delta(\omega) = \frac{1}{2\pi}\delta(f)$$

  and

$$\delta(\omega - \omega_0) = \delta(2\pi f - 2\pi f_0) = \frac{1}{2\pi}\delta(f - f_0)$$

  where $\omega = 2\pi f$ and $\omega_0 = 2\pi f_0$.

- $\delta(t-t_1) * \delta(t-t_2) = \delta\left(t-(t_1+t_2)\right)$.
- $g(t) * \delta(t-t_0) = g(t-t_0)$.
- Fourier properties:
  - Fourier series: $\delta(x-a) = \frac{1}{2\pi} + \frac{1}{\pi}\sum\limits_{k=1}^{\infty}\cos(n(x-a))$ on $[-\pi,\pi]$.
  - Fourier transform: $\delta(t) = \int 1 e^{j2\pi ft} df$
- For a function g whose real-values roots are t_i,

$$\delta\left(g\left(t\right)\right) = \sum_{k=1}^{n}\frac{\delta\left(t-t_k\right)}{\left|g'\left(t_k\right)\right|} \tag{1}$$

[2, p 387]. Similar formula also exists in probability theory[1]. Hence,

$$\int f(t)\delta(g(t))dt = \sum_{x:g(x)=0}\frac{f(x)}{|g'(x)|}. \tag{2}$$

Note that the (Dirac) delta function is to be distinguished from the *discrete time Kronecker* delta function.

As a finite measure, δ is a unit mass at 0; that is for any set A, we have $\delta(A) = 1[0 \in A]$. In which case, we have again $\int g d\delta = \int f(x)\delta(dx) = g(0)$ for any measurable g.

For a function $g : D \to \mathbb{R}^n$ where $D \subset \mathbb{R}^n$,

$$\delta(g(x)) = \sum_{z:g(z)=0}\frac{\delta(x-z)}{|\det dg(z)|} \tag{3}$$

[2, p 387].

1.2 Fourier Series

Let the (real or complex) signal $r\left(t\right)$ be a *periodic* signal with period T_0. Suppose the following ***Dirichlet*** conditions are satisfied

(a) $r\left(t\right)$ is absolutely integrable over its period; i.e., $\int\limits_0^{T_0}|r\left(t\right)|dt < \infty$.

(b) The number of maxima and minima of $r\left(t\right)$ in each period is finite.

(c) The number of discontinuities of $r\left(t\right)$ in each period is finite.

Then $r\left(t\right)$ can be expanded in terms of the complex exponential signals $\left(e^{jn\omega_0 t}\right)_{n=-\infty}^{\infty}$ as

$$\tilde{r}\left(t\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{jk\omega_0 t} + c_{-k}e^{-jk\omega_0 t}\right) \tag{4}$$

[1] In probability theory, if random variable $Y = g(X)$, the the pdf $f_Y(y) = \sum_{k=1}^{n}\frac{f_X(x_k)}{|g'(x_k)|}$.

where

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0},$$

$$c_k = \frac{1}{T_0} \int\limits_{\alpha}^{\alpha+T_0} r(t)\, e^{-jk\omega_0 t} dt, \tag{5}$$

for some *arbitrary* α. In which case,

$$\tilde{r}(t) = \begin{cases} r(t), & \text{if } r(t) \text{ is continuous at } t \\ \frac{r(t^+)+r(t^-)}{2}, & \text{if } r(t) \text{ is not continuous at } t \end{cases}$$

We give some remarks here.

- The parameter α in the limits of the integration (5) is arbitrary. It can be chosen to simplify computation of the integral. We can simply write $c_k = \frac{1}{T_0} \int\limits_{T_0} r(t)\, e^{-jk\omega_0 t} dt$ to emphasize that we only need to integrate over one period of the signal; the starting point is not important.
- The coefficients $c_k = \frac{1}{T_0} \int\limits_{T_0} r(t)\, e^{-jk\omega_0 t} dt$ are called the (k^{th}) ***Fourier (series) coefficients*** of (the signal) $r(t)$. These are, in general, complex numbers.
- $c_0 = \frac{1}{T_0} \int\limits_{T_0} r(t)\, dt$ = average or DC value of $r(t)$
- The Dirichlet conditions are only sufficient conditions for the existence of the Fourier series expansion. For some signals that do not satisfy these conditions, we can still find the Fourier series expansion.
- The quantity $f_0 = \frac{1}{T_0}$ is called the ***fundamental frequency*** of the signal $r(t)$. The nth multiple of the fundamental frequency (for positive n's) is called the nth ***harmonic***.
- $c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}$ = the k^{th} ***harmonic component*** of $r(t)$.
  $k = 1 \Rightarrow$ fundamental component of $r(t)$.
- Consider a restricted version $r_{T_0}(t)$ of $r(t)$ where we only consider $r(t)$ for one specific period. Suppose $r_{T_0}(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} R_{T_0}(f)$. Then,

  $$c_k = \frac{1}{T_0} R_{T_0}(kf_0).$$

  So, the Fourier coefficients are simply scaled samples of the Fourier transform.

1.1. Parseval's Identity: $P_r = \frac{1}{T_0} \int\limits_{T_0} |r(t)|^2 dt = \sum\limits_{k=-\infty}^{\infty} |c_k|^2$

1.2. Real, Odd/Even properties

- If $r(t)$ is even ($r(-t) = r(t)$), then $c_{-k} = c_k$.

- If $r(t)$ is odd $(r(-t) = -r(t))$, then $c_{-k} = -c_k$.
- If $r(t)$ is real valued and even, then so is c_k as a function of k.
- If $r(t)$is real-valued and odd, then c_k's are pure imaginary and $c_{-k} = -c_k$

1.3 Fourier series expansion for real valued function

Suppose $r(t)$ in the previous section is real-valued; that is $r^* = r$. Then, we have $c_{-k} = c_k^*$ and we provide here three alternative ways to represent the Fourier series expansion:

$$\tilde{r}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right) \tag{6}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(a_k \cos\left(k\omega_0 t\right)\right) + \sum_{k=1}^{\infty} \left(b_k \sin\left(k\omega_0 t\right)\right) \tag{7}$$

$$= c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos\left(k\omega_0 t + \angle c_k\right) \tag{8}$$

where the corresponding coefficients are obtained from

$$c_k = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} r(t)\, e^{-jk\omega_0 t} dt = \frac{1}{2}\left(a_k - jb_k\right) \tag{9}$$

$$a_k = 2\mathrm{Re}\left\{c_k\right\} = \frac{2}{T_0} \int_{T_0} r(t) \cos\left(k\omega_0 t\right) dt \tag{10}$$

$$b_k = -2\mathrm{Im}\left\{c_k\right\} = \frac{2}{T_0} \int_{T_0} r(t) \sin\left(k\omega_0 t\right) dt \tag{11}$$

$$|c_k| = \sqrt{a_k^2 + b_k^2} \tag{12}$$

$$\angle c_k = -\arctan\left(\frac{b_k}{a_k}\right) \tag{13}$$

$$c_0 = \frac{a_0}{2} \tag{14}$$

$$\tag{15}$$

The Parseval's identity can then be expressed as

$$P_r = \frac{1}{T_0} \int_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2$$

1.3. Examples:

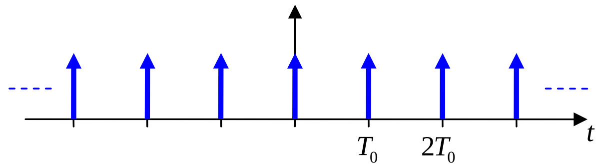

Figure 1: Train of impulses

- Train of impulses:

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - kT_0\right) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} = \frac{1}{T_0} + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos k\omega_0 t \tag{16}$$

- Square pulse periodic signal:

$$1\left[\cos\omega_0 t \geq 0\right] = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right) \tag{17}$$

We note here that multiplication by this signal is a switching function.

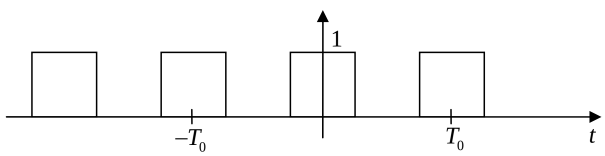

Figure 2: Square pulse periodic signal

- Bipolar square pulse periodic signal:

$$\mathrm{sgn}(\cos\omega_0 t) = \frac{4}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right)$$

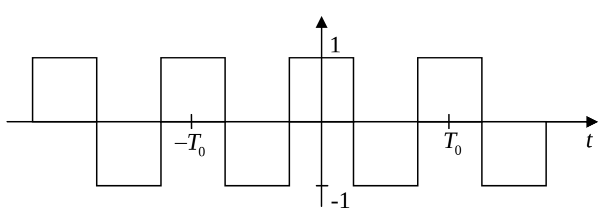

Figure 3: Bipolar square pulse periodic signal

1.4 Continuous-Time Fourier transform

The (direct) Fourier transform of $g(t)$ is defined by[2]

$$\hat{G}(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t} dt. \tag{18}$$

We may simply write $G = \mathcal{F}\{g\}$. Sometimes the magnitude and phase of G are shown explicitly by writing $G = |G|\, e^{j\theta_g}$ where both $|G|$ and θ_g are real-valued functions of ω.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(\omega)\, e^{j\omega t} d\omega = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t} dt \tag{19}$$

In MATLAB, these identities are given by `fourier` and `ifourier`. Note also that $\hat{G}(0) = \int g(t)dt$ and $g(0) = \frac{1}{2\pi}\int G(\omega)dt$.

1.4. The Fourier transform can be thought of as a continuous form of Fourier series.

1.5. Conjugate and Time Inversion (Time Reversal):

$$g(-t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}(-\omega)$$
$$g^*(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}^*(-\omega)$$
$$g^*(-t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}^*(\omega)$$

1.6. Shifting properties

- Time-shift: $g(t - t_1) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{-j\omega t_1} G(\omega)$
- Frequency-shift (or modulation): $e^{j\omega_1 t} g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}(\omega - \omega_1)$

1.7. Unit impulse:

$$e^{j\omega_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi\delta(\omega - \omega_0) = \delta(f - f_0) \tag{20}$$

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0) \tag{21}$$

$$\delta(t - t_0) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{-j\omega t_0} \tag{22}$$

$$\delta(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1 \tag{23}$$

$$1 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi\delta(\omega) \tag{24}$$

$$a \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} a2\pi\delta(\omega) \tag{25}$$

[2] MATLAB uses this definition.

Property (21) is of importance because it shows transform of periodic signal which is expressed in its Fourier series form (as in (4)). A special case is when the signal is a train of impulses which we can combine (21) and (16) to get

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \quad \text{where } \omega_0 = \frac{2\pi}{T_0}.$$

1.8. Linearity: $c_1 g_1(t) + c_2 g_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} c_1 \hat{G}_1(\omega) + \hat{G}_2(\omega)$.

1.9. Time-scaling rule: $g(at) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{|a|} \hat{G}\left(\frac{\omega}{a}\right)$

- Generally speaking, the more concentrated $g(t)$ is, the more spread out its Fourier transform $\hat{G}(\omega)$ must be.
- If we "squeeze" a function in t, its Fourier transform "stretches out" in ω. It is not possible to arbitrarily concentrate both a function and its Fourier transform.
- This trade-off can be formalized in the form of an *uncertainty principle*. See also 1.10 and 1.21.

1.10. (Heisenberg) **Uncertainty Principle** [1, 5]: Suppose g is a function which satisfies the normalizing condition $\|g\|_2^2 = \int |g(t)|^2 dt = 1$ which automatically implies that $\|G\|_2^2 = \int |G(f)|^2 df = 1$. Then

$$\left(\int t^2 |g(t)|^2 dt \right) \left(\int f^2 |G(f)|^2 df \right) \geq \frac{1}{16\pi^2}, \tag{26}$$

and equality holds if and only if $g(t) = Ae^{-Bt^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$.

- In fact, we have

  $$\left(\int t^2 |g(t - t_0)|^2 dt \right) \left(\int f^2 |G(f - f_0)|^2 df \right) \geq \frac{1}{16\pi^2},$$

  for every t_0, f_0.
- The proof relies on Cauchy-Schwarz inequality.
- For any function h, define its dispersion Δ_h as $\frac{\int t^2 |h(t)|^2 dt}{\int |h(t)|^2 dt}$. Then, we can apply (26) to the function $g(t) = h(t)/\|h\|_2$ and get

  $$\Delta_h \times \Delta_H \geq \frac{1}{16\pi^2}.$$

1.11. $\mathrm{Re}\{g(t)\} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(\hat{G}(\omega) + \hat{G}^*(-\omega)\right)$

1.12. Convolution

- Convolution-in-time Rule: $g_1(t) * g_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}_1(\omega) \cdot \hat{G}_2(\omega)$.

- Convolution-in-frequency Rule: $g_1(t) \cdot g_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2\pi}\hat{G}_1(\omega) * \hat{G}_2(\omega)$. See also (41).

1.13. Duality: Suppose $f(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} g(\omega)$. Then, $g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi f(-\omega)$.

1.14. Parseval's theorem (Plancherel formula): The total energy contained in a waveform $g(t)$ summed across all of time t is equal to the total energy of the waveform's Fourier Transform $G(f)$ summed across all of its frequency components f:

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 df \tag{27}$$

or, equivalently,

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\hat{G}(\omega)\right|^2 d\omega$$

In fact, Fourier transform preserves the inner product [1, Theorem 2.12]:

$$\langle g_1, g_2 \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) df = \langle G_1, G_2 \rangle.$$

1.15. Unit step function:

$$u(t) = 1[t \geq 0] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{j\omega} + \pi\delta(\omega) \tag{28}$$

$$\operatorname{sgn} t = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{2}{j\omega} \tag{29}$$

$$\frac{j}{\pi t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \operatorname{sgn}(\omega) \tag{30}$$

$$(g * u)(t) = \int_{-\infty}^{t} g(\tau) d\tau \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{j\omega} G(\omega) + \pi G(0)\delta(\omega) \tag{31}$$

So, if g (or equivalently, G) is band-limited to $|\omega| \leq B$, then $g * u$ is also bandlimited to $|\omega| \leq B$.

Use `heaviside` in MATLAB for $u(t)$. For example, (28) can be found by `syms t; fourier(heaviside(t))` and (32) by `syms a t; fourier( exp(-a*t)*heaviside(t))`.

1.16. Exponential: Assume $\alpha, \sigma > 0$.

$$e^{-\alpha t}u(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{\alpha + j\omega} \tag{32}$$

$$e^{\alpha t}u(-t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{\alpha - j\omega} \tag{33}$$

$$e^{-\alpha|t|} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{2\alpha}{\alpha^2 + \omega^2} \tag{34}$$

$$te^{-\alpha t}u(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{(\alpha + j\omega)^2} \tag{35}$$

$$t^n e^{-\alpha t}u(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{n!}{(\alpha + j\omega)^{n+1}} \tag{36}$$

$$ke^{-\alpha t^2} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \left(k\sqrt{\frac{\pi}{\alpha}}\right) e^{-\left(\frac{1}{4\alpha}\right)\omega^2} \tag{37}$$

$$e^{-\left(\frac{t}{\sigma}\right)^2} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \sigma\sqrt{\pi}e^{-\frac{1}{4}(\omega\sigma)^2} \tag{38}$$

1.17. Modulation:

$$\cos(\omega_c t + \theta) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \pi\delta(\omega - \omega_c)e^{j\theta} + \pi\delta(\omega + \omega_c)e^{-j\theta}$$

$$\sin(\omega_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{\pi}{j}\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$

$$g_{\omega_c,\theta}(t) = g(t) \times \cos(\omega_c t + \theta) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\hat{G}(\omega - \omega_c)e^{j\theta} + \frac{1}{2}\hat{G}(\omega + \omega_c)e^{-j\theta}$$

$$g(t) \times \sin(\omega_c t + \theta) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2j}\hat{G}(\omega - \omega_c)e^{j\theta} - \frac{1}{2j}\hat{G}(\omega + \omega_c)e^{-j\theta}$$

Suppose g is bandlimited; that is $G = 0$ for $|\omega| > \omega_g = 2\pi B_g$. If $\omega_c > \omega_g$, then the support sets of $\hat{G}(\omega - \omega_c)$ and $\hat{G}(\omega + \omega_c)$ are disjoint, and hence they are orthogonal in the frequency domain and their energy added. In which case, $E_{g_{\omega_c,\theta}} = \frac{1}{2}E_g$.

1.18. Rectangular and Sinc: Assume $a, \omega_0 > 0$.

$$1\left[|t| \le a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{\sin(2\pi f a)}{\pi f} = \frac{2\sin(a\omega)}{\omega} = 2a\ \operatorname{sinc}(a\omega),$$

$$\frac{\omega_0}{\pi}\operatorname{sinc}(\omega_0 t) = \frac{\sin(\omega_0 t)}{\pi t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1\left[|\omega| \le \omega_0\right].$$

Note that we can get a triangle from convolution of two identical rectangular waves. In particular,

$$1[|t| \le a] * 1[|t| \le a] = (2a - |t|) \times 1[|t| \le 2a].$$

Therefore,

$$\left(1 - \frac{1}{b}|t|\right)1[|t| \le b] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} b\left(\frac{\sin \pi f b}{\pi f b}\right)^2.$$

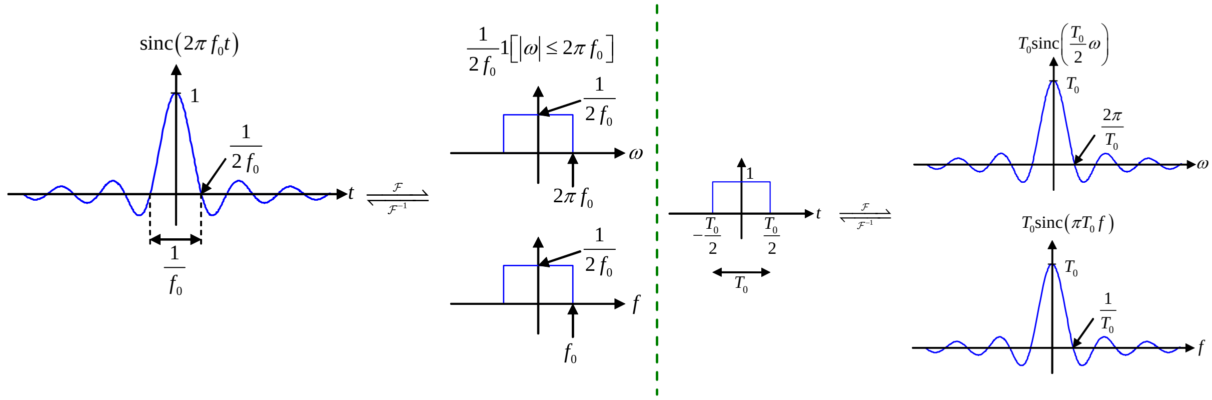

Figure 4: Fourier transform of sinc and rectangular functions

1.19. Derivative rules:

- Time-derivative rule: $\frac{d}{dt}g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} j\omega\hat{G}\left(\omega\right)$
- Frequency-derivative rule: $-jtg\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{d}{d\omega}\hat{G}\left(\omega\right)$

1.20. Real, Odd, and Even

- Conjugate Symmetry Property: If $g\left(t\right)$ is real-valued ($g\left(t\right) = g^*\left(t\right)$), then $\hat{G}\left(-\omega\right) = \hat{G}^*\left(\omega\right)$. In particular, $|G|$ is even and θ_g is odd.
- If $g\left(t\right)$ is even ($g\left(t\right) = g\left(-t\right)$), then $\hat{G}\left(\omega\right)$ is also even ($\hat{G}\left(-\omega\right) = \hat{G}\left(\omega\right)$)
- If $g\left(t\right)$ is odd ($g\left(t\right) = -g\left(-t\right)$), then $\hat{G}\left(\omega\right)$ is also odd ($\hat{G}\left(-\omega\right) = -\hat{G}\left(\omega\right)$)
- If $g\left(t\right)$ is real and even, then so is $\hat{G}\left(\omega\right)$.
- If $g\left(t\right)$ is real and odd, then $\hat{G}\left(\omega\right)$ is pure imaginary and odd.

1.21. A signal cannot be simultaneously time-limited and band-limited.

Proof. Suppose $g(t)$ is simultaneously (1) time-limited to T_0 and (2) band-limited to B. Pick any positive number T_s and positive integer K such that $f_s = \frac{1}{T_s} > 2B$ and $K > \frac{T_0}{T_s}$. The sampled signal $g_{T_s}(t)$ is given by

$$g_{T_s}(t) = \sum_k g[k]\delta\left(t - kT_s\right) = \sum_{k=-K}^{K} g[k]\delta\left(t - kT_s\right)$$

where $g[k] = g\left(kT_s\right)$. Now, because we sample the signal faster than the Nyquist rate, we can reconstruct the signal g by producing $g_{T_s} * h_r$ where the LPF h_r is given by

$$H_r(\omega) = T_s 1[\omega < 2\pi f_c]$$

with the restriction that $B < f_c < \frac{1}{T_s} - B$. In frequency domain, we have

$$G(\omega) = \sum_{k=-K}^{K} g[k]e^{-jk\omega T_s}H_r(\omega).$$

Consider ω inside the interval $I = (2\pi B, 2\pi f_c)$. Then,

$$0 \overset{\omega>2\pi B}{=} G(\omega) \overset{\omega<2\pi f_c}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) e^{-jk\omega T_s} \overset{z=e^{j\omega T_s}}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) z^{-k} \tag{39}$$

Because $z \neq 0$, we can divide (39) by z^{-K} and then the last term becomes a polynomial of the form

$$a_{2K} z^{2K} + a_{2K-1} z^{2K-1} + \cdots + a_1 z + a_0.$$

By fundamental theorem of algebra, this polynomial has only finitely many roots– that is there are only finitely many values of $z = e^{j\omega T_s}$ which satisfies (39). Because there are uncountably many values of ω in the interval I and hence uncountably many values of $z = e^{j\omega T_s}$ which satisfy (39), we have a contradiction. □

1.22. Fourier transform of periodic signal: For any periodic signal r(t) with period T_0, using the Fourier series, we can express it as

$$\tilde{r}\left(t\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}\right), \tag{4}$$

where $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$. Hence, the fourier transform is

$$R\left(f\right) = \sum_{n=-\infty}^{\infty} c_n \delta\left(f - nf_0\right)$$

1.23. Sometimes, the Fourier transform above is denoted by $\hat{G}\left(\Omega\right)$ to distinguish it from the DTFT. In which case,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}\left(\Omega\right) e^{j\Omega t} d\Omega = g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}\left(\Omega\right) = \int_{-\infty}^{\infty} g\left(t\right) e^{-j\Omega t} dt.$$

Some references define

$$G\left(f\right) = \int_{-\infty}^{\infty} g\left(t\right) e^{-j2\pi f t} dt. \tag{40}$$

In which case, we have

$$\int_{-\infty}^{\infty} G\left(f\right) e^{j2\pi f t} df = g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G\left(f\right) = \int_{-\infty}^{\infty} g\left(t\right) e^{-j2\pi f t} dt.$$

This definition eliminates several extra π and 2π factors in the identities resulting from the definition (18). In particular, there is no factor of $\frac{1}{2\pi}$ in the convolution-in-frequency formula

$$g_1\left(t\right) \cdot g_2\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_1\left(f\right) * G_2\left(f\right) \tag{41}$$

and the Parseval's theorem (27). Of course, (18) and (40) are related by

$$G\left(f\right) = \left.\hat{G}\left(\Omega\right)\right|_{\Omega=2\pi f} \quad \text{and} \quad \hat{G}\left(\Omega\right) = \left.G\left(f\right)\right|_{f=\frac{\Omega}{2\pi}}.$$

1.24. In some references, e.g. [1], the factor $1/(2\pi)$ in (19) is split between both sides of the identity resulting in a different transform pair:

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\hat{G}\left(\omega\right)e^{j\omega t}d\omega = g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}\left(\omega\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}g\left(t\right)e^{-j\omega t}dt$$

2 Energy Signal and Power Signal

For a signal $g(t)$, the instantaneous power $p(t)$ dissipated in the 1-Ω resister is $p_g(t) = |g(t)|^2$ regardless of whether $g(t)$ represents a voltage or a current. To emphasize the fact that this power is based upon unity resistance, it is often referred to as the normalized power. The total energy of the signal $g(t)$ is then

$$E_g = \int |g(t)|^2\, dt$$

and the average power is given by

$$P_g = \lim_{T\to\infty}\frac{1}{T}\int\limits_{-T/2}^{T/2}|g\left(t\right)|^2 dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\, dt.$$

If E_g is finite and nonzero, g is referred to as an ***energy signal***. If p_g is finite and nonzero, g is referred to as a ***power signal***. Note that the power signal has infinite energy and an energy signal has zero average power; thus the two categories are mutually exclusive.

2.1 Energy Signal

2.1. Definitions

- Energy: $E_g = \int |g(t)|^2\, dt$.
- Energy spectral density (ESD): $\Psi_g(t) = |G(\omega)|^2$.
    - ESD is a positive, real, and even function of ω.
- Time autocorrelation function:

$$\begin{aligned}\psi_g(\tau) &= \int g^*(\mu)g(\mu+\tau)d\mu = g^*(\tau) * g(-\tau) \\ &= \int g(\mu)g^*(\mu-\tau)d\mu = g(\tau) * g^*(-\tau)\end{aligned}$$

    - ψ_g is invariant to time-shift in g: Suppose $h(t) = g(t-t_0)$, then $\psi_g = \psi_h$.

2.2.

$$E_g = \int |g(t)|^2 \, dt$$
$$= \frac{1}{2\pi} \int |G(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int \Psi_g(\omega) d\omega = \int \Psi_g(2\pi f) df$$

2.3. $\psi_g(\tau) \xrightarrow{\mathcal{F}} \Psi_g(\omega)$

2.4. Example

- For $g(t) = 1_{[t_0, t_0+T]}(t)$, we have $\psi_g(\tau) = \left(1 - \frac{|\tau|}{T}\right) 1_{[-T,T]}(\tau)$ and $\Psi_g(\omega) = T \operatorname{sinc}^2\left(\frac{\omega T}{2}\right)$.

2.5. Suppose g and y are the input and output signals of an LTI system with transfer function H $\left(g \to \boxed{\text{LTI} : H} \to y\right)$, then $\Psi_y(\omega) = |H(\omega)|^2 \Psi_g(\omega)$.

2.2 Power Signal

2.6. Definitions:

- $g_T(t) = g(t) 1\left[|t| \le \frac{T}{2}\right]$
- Power: $P_g = \lim\limits_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} |g(t)|^2 dt = \lim\limits_{T\to\infty} \frac{1}{T} E_{g_T} = \langle g^2 \rangle$.
- Power spectral density (PSD): $S_g(\omega) = \lim\limits_{T\to\infty} \frac{1}{T} |G_T(\omega)|^2 = \lim\limits_{T\to\infty} \frac{1}{T} \Psi_{g_T}^2(t)$
  - PSD represents the power per unit bandwidth (in Hz) of the spectral components at the frequency ω.
  - PSD is a positive, real, and even function of ω.
- Time autocorrelation function:

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} g^*(\mu) g(\mu + \tau) d\mu = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} g(\mu) g^*(\mu - \tau) d\mu$$
$$= \langle g^*(\cdot) g(\cdot + \tau) \rangle = \langle g(\cdot) g^*(\cdot - \tau) \rangle$$

  - $R_g(-\tau) = R_g^*(\tau)$

2.7.

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} E_{g_T} = \left\langle g^2 \right\rangle$$
$$= \frac{1}{2\pi} \int S_g(\omega) d\omega = \int S_g(2\pi f) df$$

2.8. $R_g(\tau) \xrightarrow{\mathcal{F}} S_g(\omega)$

2.9. $R_{g_1 g_2}(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} g_1(t) g_2(t+\tau) dt$

- If $g = g_1 + g_2$, then $R_g = R_{g_1} + R_{g_2} + R_{g_1 g_2} + R_{g_2 g_1}$.

2.10. Examples

- $g(t) = a\cos(\omega_0 t + \theta)$
  - $R_g(\tau) = \frac{1}{2}a^2 \cos\omega_0 t$
  - $S_g(\omega) = \frac{\pi}{2}a^2 (\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$
- Periodic function $r(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t + \theta_n)$
  - $R_r(\tau) = d_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} d_n^2 \cos n\omega_0 \tau$
  - $S_r(\omega) = 2\pi d_0^2 \delta(\omega) + \frac{\pi}{2}\sum_{n=1}^{\infty} d_n^2 (\delta(\omega - n\omega_0) + \delta(\omega + n\omega_0))$

2.11. Suppose g and y are the input and output signals of an LTI system with transfer function H $\left(g \to \boxed{\text{LTI} : H} \to y\right)$, then $S_y(\omega) = |H(\omega)|^2 S_g(\omega)$.

3 Modulation

Let the ***carrier frequency*** be at f_c [Hz] with corresponding angular frequency $\omega_c = 2\pi f_c$.

3.1. ***Double-sideband suppressed carrier (DSB-SC)*** modulation:

(a) See Figure 6.

(b) Modulation

(i) Modulated signal: $x(t) = m(t)\cos\omega_c t$

(ii) Recall from (1.17) that

$$g_{\omega_c,\theta}(t) = g(t) \times \cos(\omega_c t + \theta) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\hat{G}(\omega - \omega_c) e^{j\theta} + \frac{1}{2}\hat{G}(\omega+\omega_c) e^{-j\theta}.$$

Furthermore, if (1) g is bandlimited to $|\omega| \le \omega_g = 2\pi B_g$ and (2) $|\omega| > \omega_g = 2\pi B_g$, then $E_{g_{\omega_c,\theta}} = \frac{1}{2}E_g$. This is not true for non-bandlimited g. For example, take $g = 1_{[0,T]}$, then

$$\int g^2(t)\cos^2(\omega_c t) dt = \frac{E_g}{2} + \frac{1}{2\omega}\cos(\omega_c T)\sin(\omega_c T)$$

where the second term does not vanish for all ω_c. It will vanish when $\omega_c \to \infty$.

(iii) To produce the modulated signal $m(t)\cos\omega_c t$, we may use the following methods which generate the modulated signal along with other signals which can be eliminated by a bandpass filter restricting frequency contents to around ω_c.

i. When it is easier to build a squarer than a multiplier, use

$$\begin{aligned}(m(t) + c\cos(\omega_c t))^2 &= m^2(t) + c^2\cos^2(\omega_c t) + 2cm(t)\cos(\omega_c t) \\ &= m^2(t) + \frac{c^2}{2} + 2cm(t)\cos(\omega_c t) + \frac{c^2}{2}\cos(2\omega_c t).\end{aligned}$$

Alternative, can use $\left(m(t) + c\cos\left(\frac{\omega_c}{2}t\right)\right)^3$.

ii. Multiply $m(t)$ by "any" periodic and even signal $r(t)$ whose period is $T_c = \frac{2\pi}{\omega_c}$. Because $r(t)$ is an even function, we know that

$$r(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_c t).$$

Therefore,

$$m(t)r(t) = c_0 m(t) + \sum_{k=1}^{\infty} a_k m(t)\cos(k\omega_c t).$$

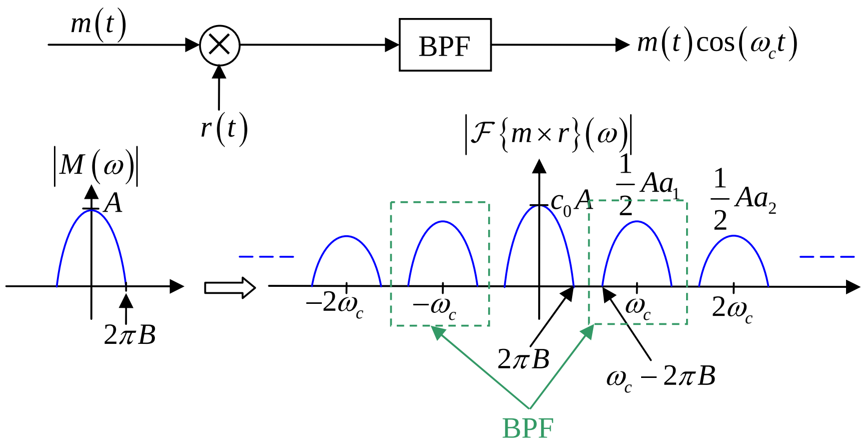

Figure 5: Modulation of $m(t)$ via even and periodic $r(t)$

See also [3, p 157]. In general, for this scheme to work, we need

- $a_1 \neq 0$; that is T_c is the "least" period of r;
- $\omega_c > 4\pi B$; that is $f_c > 2B$ (to prevent overlapping).

Note that if $r(t)$ is not even, then by (8), the outputted modulated signal is of the form $a_1 m(t)\cos(\omega_c t + \phi_1)$.

iii. ***Switching modulator***: set $r(t)$ to be the square pulse train given by (17). Multiplying this $r(t)$ to the signal $m(t)$ is equivalent to switching $m(t)$ on and off periodically.
It is equivalent to periodically turning the switch on (letting $m(t)$ pass through) for half a period $T_c = \frac{1}{f_c}$.

(iv) Need $\omega_c \geq 2\pi B$

(v) The modulated signal spectrum centered at ω_c is composed of two parts: a portion that lies above ω_c, known as the upper sideband (USB), and a portion that lies below ω_c, known as the lower sideband (LSB). Hence, this is a modulation scheme with double sidebands.

(vi) The modulated signal does not contain a discrete component of the carrier frequency ω_c.

(c) Demodulation:

(i) Basic idea:

$$\mathrm{LPF}\left\{\left(m\left(t\right)\sqrt{2}\cos\omega_c t\right)\sqrt{2}\cos\left(\left(\omega_c+\Delta\omega\right)t+\theta\right)\right\}=m\left(t\right)\cos\left(\left(\Delta\omega\right)t+\theta\right).$$

Of course, we want $\Delta\omega = 0$ and $\theta = 0$; that is the receiver must generate a carrier in phase and frequency synchronism with the incoming carrier. These demodulators are called ***synchronous*** or ***coherent*** (also ***homodyne***) demodulator [3, p 161].

(ii) Suppose the propagation time is τ, then we have

$$\mathrm{LPF}\left\{\left(m\left(t-\tau\right)\sqrt{2}\cos\left(\omega_c\left(t-\tau\right)\right)\right)\sqrt{2}\cos\left(\omega_c\left(t-\mu\right)\right)\right\}=m\left(t-\tau\right)\cos\left(\omega_c\left(\tau-\mu\right)\right)$$

At the receiver, we want $\mu = \tau$.

(iii) ***Envelope detector***. See [3, p 168]. Note that this method need $m(t) \geq 0$.

(iv) ***Switching Demodulator***:

$$\mathrm{LPF}\{m(t)\cos(\omega_c t)\times 1[\cos(\omega_c t)\geq 0]\}=\frac{1}{\pi}m(t) \tag{42}$$

[3, p 162].

(v) ***Rectifier Detector***: Suppress the negative part of $m(t)cos(\omega_c t)$ using a diode. This is equivalent to switching demodulator in (42). It is in effect synchronous detection performed without using a local carrier [3, p 167].

3.2. *Amplitude Modulation* (*AM*):

$$\varphi_{\mathrm{AM}}\left(t\right)=\left(A+m\left(t\right)\right)\cos\omega_c t=\underbrace{A\cos\omega_c t}_{\text{carrier}}+\underbrace{m\left(t\right)\cos\omega_c t}_{\text{sidebands}}$$

3.3. *Quadrature amplitude modulation* (*QAM*):

$$\varphi_{\mathrm{QAM}}\left(t\right)=m_1\left(t\right)\cos\left(\omega_c t\right)+m_2\left(t\right)\sin\left(\omega_c t\right)$$

In complex form,

$$\varphi_{\mathrm{QAM}}\left(t\right)=\mathrm{Re}\left\{\left(m_1(t)-jm_2(t)\right)e^{-j\omega_c t}\right\}.$$

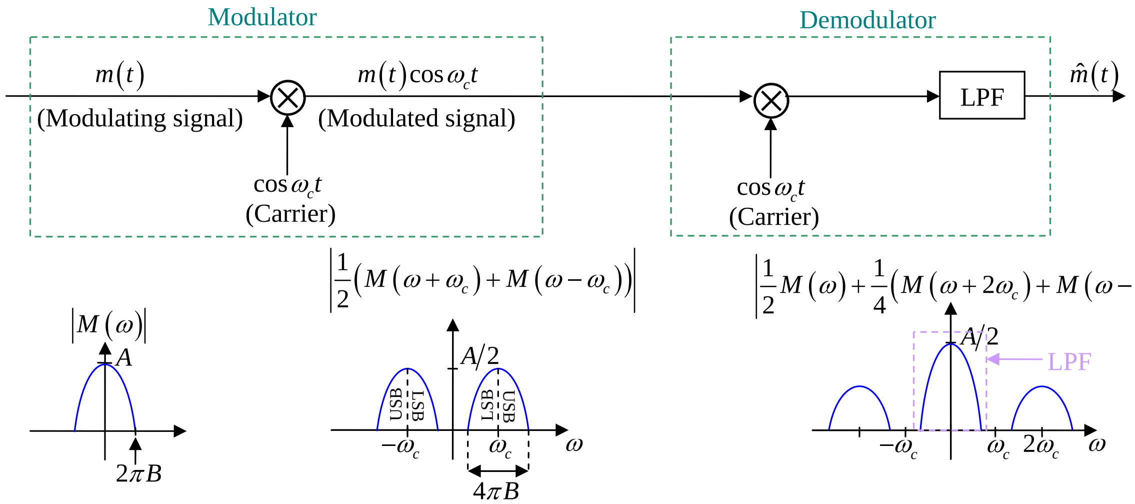

Figure 6: DSB-SC modulation and demodulation

At the receiver, when there is some frequency offset and phase shift:

$$\text{LPF}\left\{\varphi_{\text{QAM}}(t)\,2\cos\left((\omega_c+\Delta\omega)\,t+\delta\right)\right\} = m_1(t)\cos\left((\Delta\omega)\,t+\delta\right) - m_2(t)\sin\left((\Delta\omega)\,t+\delta\right)$$
$$\text{LPF}\left\{\varphi_{\text{QAM}}(t)\,2\sin\left((\omega_c+\Delta\omega)\,t+\delta\right)\right\} = m_1(t)\sin\left((\Delta\omega)\,t+\delta\right) + m_2(t)\cos\left((\Delta\omega)\,t+\delta\right)$$

Definition 3.4 (Instantaneous frequency)**.** Consider a ***generalized sinusoidal*** signal

$$x(t) = A\cos\theta(t)$$

where $\theta(t)$ is the ***generalized angle***.

- The generalized angle for conventional sinusoid is $\omega_c t + \theta_0$.

Define the ***instantaneous frequency*** ω_i at t to be the slope of $\theta(t)$ at t; that is

$$\omega_i(t) = \frac{d}{dt}\theta(t).$$

Therefore,

$$\theta(t) = \int_{-\infty}^{t} \omega_i(\tau)d\tau. \tag{43}$$

- It is tempting to use $\tilde{x}(t) = A\cos(\omega_i(t)t+\theta_0)$ instead of $A\cos\theta(t)$ given by (43). The idea is that we replace the frequency term in the standard sinusoid with the instantaneous frequency. This can lead to very different results. In particular, the instantaneous frequency of $\tilde{x}$ is $\omega_i(t) + \omega_i'(t)\,t \neq \omega_i(t)$.

  For example, suppose the instantaneous frequency is given by $\omega_i(t) = t$. Then, the instantaneous frequency of $\tilde{x}$ is $2t$ which doubles the desired frequency.

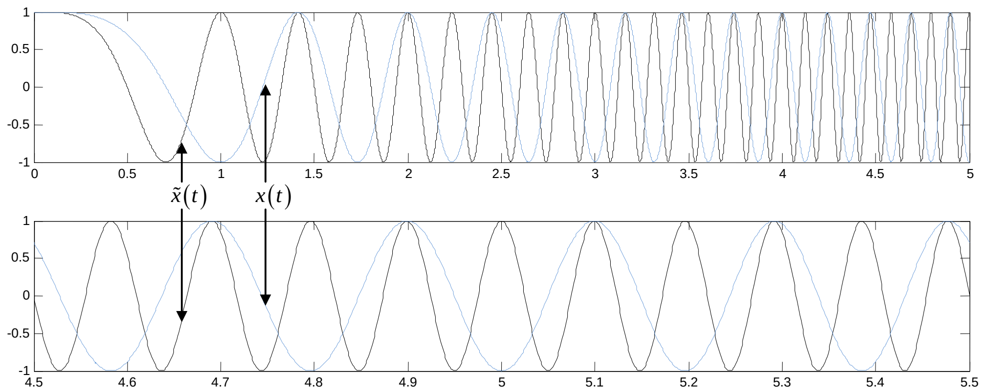

Figure 7: Comparison between $\tilde{x}(t) = \cos(t \times t)$ and $x(t) = \cos\left(\int_0^t \tau d\tau\right)$. The "frequencies" at $t = 5$ of x and $\tilde{x}$ are 5 and 10, respectively.

3.5. ***Angle modulation*** (or ***exponential modulation***)

(a) ***Generalized angle modulation***:

$$\varphi(t) = A\cos\left(\omega_c t + \theta_0 + (m * h)(t)\right)$$

where h is causal.

(b) ***Frequency modulation*** (***FM***):

$$\varphi_{\mathrm{FM}}(t) = A\cos\left(\omega_c t + \theta_0 + k_f \int_{-\infty}^{t} m(\tau) d\tau\right).$$

- The instantaneous frequency is given by

$$\omega_i(t) = \omega_c + k_f m(t).$$

- $h(t) = k_f 1_{[0,\infty)}(t)$
- The BW is $\approx 2k_f m_p$ where m_p is the peak amplitude of $m(t)$.

(c) ***Phase modulation*** (***PM***):

$$\varphi_{\mathrm{PM}}(t) = A\cos\left(\omega_c t + \theta_0 + k_p m(t)\right)$$

- $h(t) = k_p \delta(t)$.
- PM is actually the FM when modulating signal is $m'(t)$.
- The BW is $\approx 2k_f m_p'$ where m_p' is the peak amplitude of $m'(t)$. (To see this, use the above observation and the approximation for the BW of FM.)

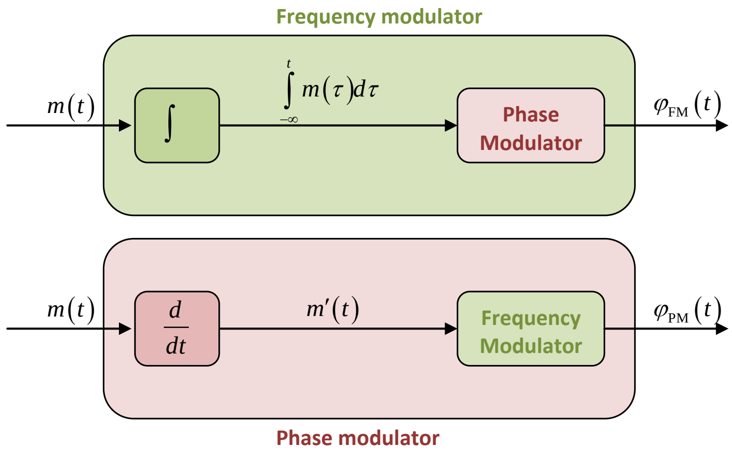

Figure 8: PM and FM are inseparable [3, Fig 5.2].

4 Sampling Theorem

A low-pass signal g whose spectrum is band-limited to B Hz ($G(\omega) = 0$ for $|\omega| > 2\pi B$) can be reconstructed exactly (without any error) from its sample taken uniformly at a rate (sampling frequency) $R_s > 2B$ Hz (samples per second).

4.1. The "sampling" can be done by producing

$$g_{T_s}(t) = g(t)\, r_{T_s}(t)$$

where r_{T_s}

(a) is periodic with period $T_s = \frac{1}{R_s} < \frac{1}{2B}$

(b) has nonzero mean.

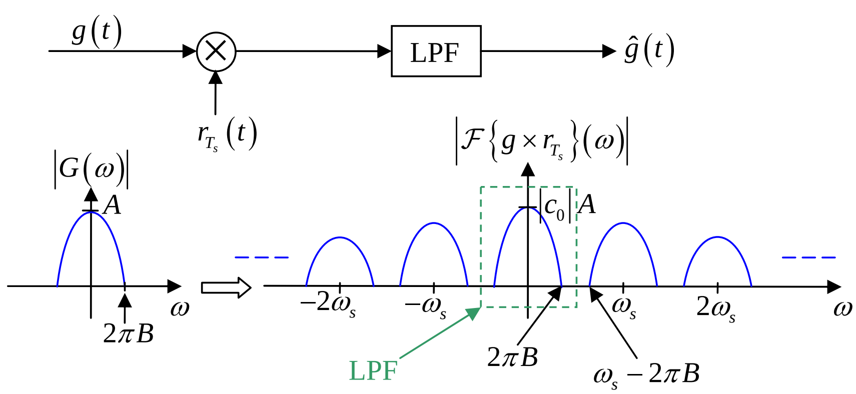

Figure 9: Sampling and Reconstruction

4.2. ***Signal Reconstruction***: Because r_{T_s} is periodic, it has fourier series expansion

$$\tilde{r}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where $\omega_s = 2\pi f_s = \frac{2\pi}{T_s}$. Hence,

$$G_{T_s}(\omega) = \sum_n c_n G(\omega - n\omega_s).$$

Suppose $2\pi B < \omega_s - 2\pi B$ (or equivalently $R_s > 2B$), then there is no overlapping and we can get G back by LPF H with cutoff $f_c \in [B, R_s - B)$. More specifically,

$$H(\omega) = \frac{1}{c_0} 1[|\omega| \le \omega_c] \underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftharpoons}} h(t) = \frac{\omega_c}{c_0 \pi} \operatorname{sinc}(\omega_c t) = \frac{2f_c}{c_0} \operatorname{sinc}(2\pi f_c t).$$

4.3. ***Interpolation formula***: Suppose r_{T_s} is a train of impulses δ_{T_s} as in (16). In which case,

$$g_{T_s}(t) = \sum_k g[k] \delta(t - kT_s)$$

where $g[k] = g(kT_s)$. Note that we have $c_0 = \frac{1}{T_s} = f_s$. Therefore,

$$H(\omega) = T_s 1[|\omega| \le \omega_c] \underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftharpoons}} h(t) = \frac{2f_c}{f_s} \operatorname{sinc}(2\pi f_c t).$$

The filtered output $\hat{g} = g_{T_s} * h$ which is g can now be expressed as a sum

$$g(t) = \sum_k g[k] h(t - kT_s) = \frac{2f_c}{f_s} \sum_k g[k] \operatorname{sinc}(2\pi f_c (t - kT_s))$$

Furthermore, suppose we choose $f_s = 2B$ and $f_c = B$, then we have

$$H(\omega) = \frac{1}{2B} 1[|\omega| \le 2\pi B] \underset{\mathcal{F}}{\overset{\mathcal{F}^{-1}}{\rightleftharpoons}} h(t) = \operatorname{sinc}(2\pi B t).$$

In which case,

$$g(t) = \sum_k g[k] \operatorname{sinc}(2\pi B (t - kT_s)) = \sum_k g[k] \operatorname{sinc}(2\pi B t - k\pi).$$

4.4. A maximum of $2B$ independent pieces (samples/symbols) of information per second can be transmitted, errorfree, over a noiseless channel of bandwidth B Hz [3, p 260].

- Start with $2B$ pieces of information per second. Denote the sequence of such information by m_k.
- Construct a signal $g(t)$ whose (Nyquist) sample values $g\left(k\frac{1}{2B}\right)$ agrees with m_k.

4.5. A band pass signal whose spectrum exists over a frequency band $f_c - \frac{B}{2} < |f| < f_c + \frac{B}{2}$ ha s a bandwidth B Hz. Such a signal is uniquely determined by $2B$ samples per second. The sampling scheme uses two interlaced sampling trains, each at a rate of B samples per second (known as second-order sampling).

5 Discrete Fourier transform (DFT)

In DFT, we work with N-point signal (finite-length sequence of length N) in both time and frequency domain. To simplify the definition we define

$$\psi_N = e^{j\frac{2\pi}{N}}$$

and the DFT matrix $Q = \Psi_N$ whose element on the pth row and qth column is given by $\psi_N^{-(p-1)(q-1)}$:

$$\Psi_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \psi_N^{-1} & \psi_N^{-2} & \cdots & \psi_N^{-(N-1)} \\ 1 & \psi_N^{-2} & \psi_N^{-4} & \cdots & \psi_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi_N^{-(N-1)} & \psi_N^{-2(N-1)} & \cdots & \psi_N^{-(N-1)(N-1)} \end{bmatrix}$$

5.1. Properties of ψ_N:

(a) $\psi_1 = 1$ and $\psi_2 = -1$.

(b) For even n, $\psi_N^{N/2} = e^{j\pi} = -1$

(c) $(\psi_N)^N = 1$

(d) $\psi_N^* = (\psi_N)^{-1}$

(e) If $N \mod p = 0$, then $\psi_N^p = \psi_{\frac{N}{p}}$

- $\psi_{kN}^N = \psi_k$

(f) $\sum_{n=0}^{N-1} \psi_N^{-kn} = \begin{cases} N & k = mN \\ 0 & \text{otherwise} \end{cases}$

5.2. Properties of Ψ_N

- $\Psi_N^{-1} = \frac{1}{N}\Psi_N^*$. Equivalently, $\Psi_N^{-1}\Psi_N = NI_N$.

Definition 5.3. The N-point DFT of an N-point signal (column vector) x is given by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jnk\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk} \quad ; 0 \le k < N.$$

The inverse DFT is given by

$$\underset{0\le n<N}{x[n]} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, \psi_N^{nk} \underset{\text{DFT}^{-1}}{\overset{\text{DFT}}{\rightleftharpoons}} \underset{0\le k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk}$$

In matrix form,

$$x = \frac{1}{N}\Psi_N^* X \underset{\text{DFT}^{-1}}{\overset{\text{DFT}}{\rightleftharpoons}} X = \Psi_N \times x.$$

- $\delta[n-n_0] \xrightleftharpoons[\text{DFT}^{-1}]{\text{DFT}} \psi_N^{-n_0 k}$, $0 \le n_0 < N$.

5.4. N-Point Circular Convolution

$$\{x_1 \circledast x_2\}[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2[(n-m) \bmod N] = \sum_{m=0}^{N-1} x_1[(n-m) \bmod N]\, x_2[m]$$

- Example: For $N = 3$, if $y = x_1 \circledast x_2$, then

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix} = \begin{bmatrix} x_1[0]x_2[0] + x_1[1]x_2[\langle -1\rangle_3] + x_1[2]x_2[\langle -2\rangle_3] \\ x_1[0]x_2[1] + x_1[1]x_2[\langle 0\rangle_3] + x_1[2]x_2[\langle -1\rangle_3] \\ x_1[0]x_2[2] + x_1[1]x_2[\langle 1\rangle_3] + x_1[2]x_2[\langle 0\rangle_3] \end{bmatrix}$$
$$= \begin{bmatrix} x_1[0]x_2[0] + x_1[1]x_2[2] + x_1[2]x_2[1] \\ x_1[0]x_2[1] + x_1[1]x_2[0] + x_1[2]x_2[2] \\ x_1[0]x_2[2] + x_1[1]x_2[1] + x_1[2]x_2[0] \end{bmatrix}$$
$$= \begin{bmatrix} x_2[0] & x_2[2] & x_2[1] \\ x_2[1] & x_2[0] & x_2[2] \\ x_2[2] & x_2[1] & x_2[0] \end{bmatrix} \times \begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \end{bmatrix}$$

A Trig Identities

All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at origin as shown in Figure 10.

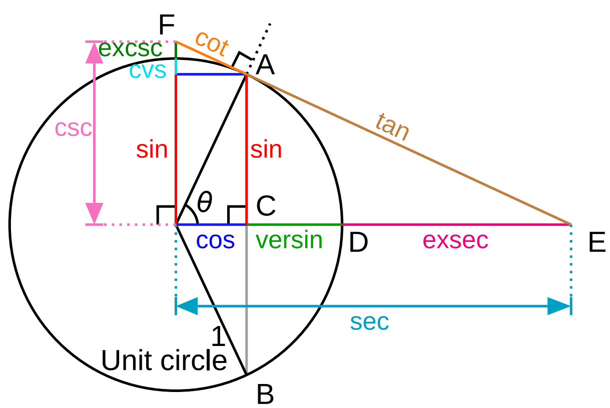

Figure 10: Trigonometric functions on a unit circle.

A.1. Cosine function

(a) Is an even function: $\cos(-x) = \cos(x)$.

(b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$.

(c) Sum formula:
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y. \tag{44}$$

(d) Product-to-Sum Formula:
$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right).$$

(e) $\cos^n x = \begin{cases} \frac{1}{2^{n-1}} \sum\limits_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos\left((n-2k)\,x\right), & \text{odd } n \geq 1 \\ \frac{1}{2^n} \left(\sum\limits_{k=0}^{\frac{n}{2}-1} 2\binom{n}{k} \cos\left((n-2k)\,x\right) + \binom{n}{\frac{n}{2}} \right), & \text{even } n \geq 2 \end{cases}$

(f) Any two real numbers a, b can be expressed in terms of cosine and sine with the same amplitude and phase:
$$(a, b) = \left(A\cos(\phi), A\sin(\phi)\right), \tag{45}$$
where $A = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}\frac{b}{a}$. This is simply the polar-coordinates from of point (a, b) on Cartesian coordinates.

A.2. Properties of e^{ix}

(a) ***Euler's formula***: $\boxed{e^{ix} = \cos x + i \sin x.}$ Hence,
$$\cos(A) = \operatorname{Re}\left(e^{jA}\right) = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$
$$\sin(A) = \operatorname{Im}\left(e^{jA}\right) = \operatorname{Re}\left(-je^{jA}\right) = \operatorname{Re}\left(-\frac{1}{j}e^{jA}\right) = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right).$$

- We can use $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ and $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$ to derive many trigonometric identities.

In fact, we can combine linear combination of cosine and sine of the same argument into a single cosine by
$$A\cos\omega_0 t + B\sin\omega_0 t = \sqrt{A^2 + B^2}\cos\left(\omega_0 t - \tan^{-1}\frac{B}{A}\right).$$

To see this, note that
$$\begin{aligned} A\cos\omega_0 t + B\sin\omega_0 t &= \operatorname{Re}\left(Ae^{j\omega_0 t}\right) + \operatorname{Re}\left(-jBe^{j\omega_0 t}\right) = \operatorname{Re}\left((A - jB)\,e^{j\omega_0 t}\right) \\ &= \operatorname{Re}\left(\sqrt{A^2 + B^2}e^{-j\tan^{-1}\frac{B}{A}}e^{j\omega_0 t}\right). \end{aligned}$$

Another way to see this is to reexpress the two real numbers A, B using (45) and then use (44).

(b) e^{jx} is periodic with period 2π.

(c) Any complex number $z = x + jy$ can be expressed as $z = \sqrt{x^2 + y^2} e^{j \tan^{-1}\left(\frac{y}{x}\right)} = |z| e^{j\phi}$.

- $z^t = |z|^t e^{j\phi t}$.

(d) More relations involving sin and cos.

- $e^{jAt} + e^{jBt} = 2e^{j\frac{A+B}{2}t} \cos\left(\frac{A-B}{2}\right)$.
- $e^{jAt} - e^{jBt} = 2je^{j\frac{A+B}{2}t} \sin\left(\frac{A-B}{2}\right)$
- $\frac{e^{jAt} - e^{jBt}}{e^{jCt} - e^{jDt}} = e^{j\frac{(A+B)-(C+D)}{2}t} \frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C-D}{2}\right)}$.

References

[1] Albert Boggess and Francis J. Narcowich. *First Course in Wavelets with Fourier Analysis*. Prentice Hall, 1 edition, 2001. 1.10, 1.14, 1.24

[2] Donald G. Childers. *Probability And Random Processes Using MATLAB*. McGraw-Hill, 1997. 1.1, 1.1

[3] B. P. Lathi. *Modern Digital and Analog Communication Systems*. Oxford University Press, 1998. 2(c)ii, 3a, 3c, 3d, 3e, 8, 4.4

[4] C. Britton Rorabaugh. *Communications Formulas and Algorithms: For System Analysis and Design*. Mcgraw-Hill, 1990.

[5] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton University Press, March 2003. 1.10