Tuesday, January 04, 2011
10:53 AM

Lecture 13 (Jan 4)

Today, we will consider a more realistic system where the infinite-user assumption in M/M/m/m is relaxed.

Erlang B via Markov chain and discrete-time approximation

Recall:

Call initiation request : Poisson process with rate λ

$\uparrow$ aggregated rate (total)

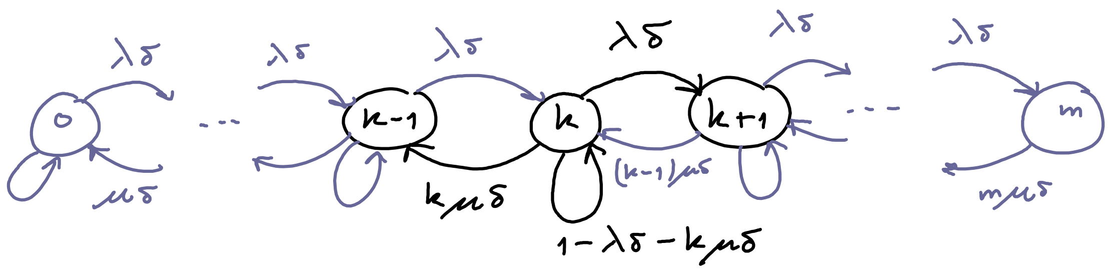
combination of rates from all users.

Assume : Infinite # users.

So, by having, e.g. 5 users talking on the phone will not change the total call request rate. (The call request rate for each user is extremely small; so missing five of them does not change the sum rate of the infinitely many users that are not making call.

Call duration $\sim \mathcal{E}(\mu)$ where $\frac{1}{\mu}$ = average call duration

The state transition diagram when # channels = m :



Note that we have $\xrightarrow{\lambda\delta}$ on the top for all the state. It does not matter how many customers are talking on the phone, the total rate is still λ.

Recall that

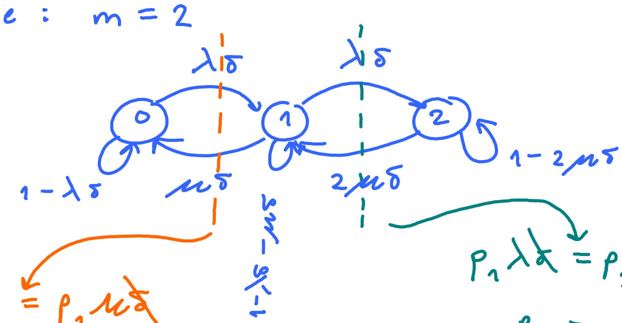
p_k = the probability that the system will be in state k (in the long run).

We can find these steady-state probabilities $p_0, p_1, \ldots, p_m$

by considering the flow of probabilities ~~across~~ boundaries.

They are usually denoted by $\pi_0, \pi_1, \ldots, \pi_m$

Example : $m = 2$



$$p_1 \lambda \cancel{\delta} = p_2 \, 2\mu \cancel{\delta}$$

$$p_2 = \frac{\lambda}{2\mu} p_1 = \frac{1}{2} A p_1$$

$$= \frac{1}{2} A \times A p_0 = \frac{1}{2} A^2 p_0$$

$$p_0 \times \lambda \cancel{\delta} = p_1 \mu \cancel{\delta}$$

$$p_1 = p_0 \frac{\lambda}{\mu} = A p_0$$

$$p_0 + p_1 + p_2 = 1$$

$$p_0 + A p_0 + \frac{1}{2} A^2 p_0 = 1 \Rightarrow p_0 = \frac{1}{1 + A + \frac{1}{2} A^2}$$

$$p_1 = A p_0 = \frac{A}{1 + A + \frac{1}{2} A^2}$$

$$p_2 = \frac{1}{2} A^2 p_0 = \frac{\frac{1}{2} A^2}{1 + A + \frac{1}{2} A^2}$$

Erlang B $(2, A)$

In general, if we have m channel,

$$p_m = \frac{\frac{A^m}{m!}}{\sum_{k=0}^{m} \frac{A^k}{k!}}$$

Erlang B (m, A)

This is the probability that the system is in state m.
When the system is in state m, all channel are used and therefore any new call request will be blocked.

It turnrout that p_m is the same as call blocking probability, which is the long-run proportion of call requests that get blocked.

(You verify this fact in HW3)

Engset Model

(Engsett)

not infinite

- Finite ~~#~~ users : N users.
- Each user generates new call request with

- Each user generates new call request with rate λ

  (might be better to call it λ_u)

  So, "total" rate $= \lambda \times N$

Same as in the derivation of Erlang B formula:

- m channels
- call duration $\sim \mathcal{E}(\mu)$ $\left(\text{average} = \frac{1}{\mu}\right)$

Remarks

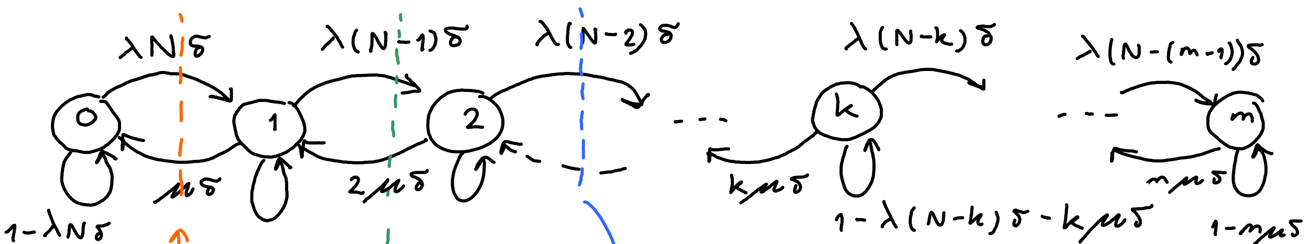
1. The call generation process for each user will <u>not</u> be a Poisson process. A user will <u>not</u> generate new call when he or she is talking on the phone. The rate λ assumed means that after he or she finish a call, the time until this user make a new call request is $\mathcal{E}(\lambda)$.

   If a user request a call but is blocked, the time until this user make a new call request is also $\mathcal{E}(\lambda)$.

2. Because a user can not generate new call when he/she is already involved in a call, if the system is in state k, there are only $N-k$ users that can generate new calls. Hence, the "total" call request rate for state k is $(N-k)\lambda$.

   This means that the $\xrightarrow{\lambda\delta}$ that we had in the derivation of the Erlang B formula will need to be changed so that their values depend on the state of the system.

Here is the new state transition diagram:



$$p_0 \times \lambda N \cancel{\delta} = p_1 \times \mu \cancel{\delta}$$

$$p_1 = p_0 \frac{\lambda N}{\mu} = p_0 A N$$

$$p_1 \times \lambda(N-1)\cancel{\delta} = p_2 \times 2\mu\cancel{\delta}$$

$$p_2 = A\frac{(N-1)}{2} p_1$$

$$= A\frac{(N-1)}{2} N A p_0$$

$$= \frac{N(N-1)}{2} A^2 p_0$$

$$p_2 \times \lambda(N-2)\cancel{\delta} = p_3 \times 3\mu\cancel{\delta}$$

$$= \frac{N(N-1)}{2} A^2 p_0 = \binom{N}{2} A^2 p_0$$

$$p_3 = \frac{\lambda}{\mu} \frac{N-2}{3} \times p_2$$

$$= A \frac{N-2}{3} \times \frac{N(N-1)}{2} A^2 p_0$$

$$= A^3 \frac{N(N-1)(N-2)}{3 \times 2 \times 1} p_0 = A^3 \binom{N}{3} p_0$$

$$\vdots$$

$$p_k = A^k \binom{N}{k} p_0 \qquad 0 \leq k \leq m$$

$$\sum_{k=0}^{m} p_k = 1 \quad \Rightarrow \quad \sum_{k=0}^{m} A^k \binom{N}{k} p_0 = 1$$

$$p_0 = \frac{1}{\sum_{k=0}^{m} \binom{N}{k} A^k}$$

Normalization factor $\rightarrow Z(m, N)$

$$p_k = A^k \binom{N}{k} p_0 = \frac{A^k \binom{N}{k}}{Z(m,N)}$$

$$p_m = \frac{A^m \binom{N}{m}}{Z(m,N)} = \boxed{\frac{\binom{N}{m} A^m}{\sum_{k=0}^{m} \binom{N}{k} A^k}}$$

Engset

← time congestion probability.

~~call blocking probability~~

Again, this is the probability (in the long run) that you will find the system in state m (where new calls will be blocked because all channels are occupied.

Unlike the Erlang B case, this number is not the same as the call blocking probability.

To find call blocking probability,

consider s slots (s is very large)

Then, about $p_k \times s$ slots will be in state k.

Each of these slots will have a new call request with probability

$(N-k)\lambda\delta$ ← Recall that $p_1 = \lambda\delta$ when we study Erlang B derivation.

\# new calls request $\approx (N-k)\lambda\delta \times (p_k \times s)$

This approximates the number of new calls when the system is in state k.

Total \# new calls request $\approx \sum_{k=0}^{m} (N-k)\lambda\delta\, p_k\, s$

\# blocked call $= (N-m)\lambda\delta \times (p_m \times s)$ ($k=m$)

This is the number of new call requests that are made when the system is in state m and hence get blocked.

Blocked call proportion (probability) $= \dfrac{(N-m)\cancel{\lambda\delta}\, p_m \times \cancel{s}}{\sum_{k=0}^{m}(N-k)\cancel{\lambda\delta}\, p_k \times \cancel{s}}$

$= \dfrac{(N-m)\,p_m}{\sum_{k=0}^{m}(N-k)\,p_k}$

call congestion (the probability that a call is blocked)

don't have (N-k) weights.

$\left(\text{For Erlang B}, = \dfrac{p_m}{\underbrace{\sum_{k=0}^{m} p_k}_{1}} = p_m \right)$

Putting the expression for p_k into the formula above, we have

Blocked call probability $= \boxed{\dfrac{(N-m)\binom{N}{m}A^m}{\sum_{k=0}^{m}(N-k)\binom{N}{k}A^k}}$

Engset formula
for call blocking probability