Tuesday, December 14, 2010
1:04 PM

Lecture 10 (Dec 14)

HW3 is posted.

Due Dec. 21 (next week) (Tuesday)

@ 2:39 PM

Midterm Exam ← Not to torture you.

Dec 24 9-12 AM

BKD 3206

Closed book / Closed Note

No cheat sheet

Basic calculator allowed

More info to be added on the course web site.

Some formulas will be provided on the exam paper.

↑ will be posted this weekend

Poisson Process (PP)

⇒ Random arrangement of "marks" (denoted by "x") on the time line.

"marks": arrival times / request times

Only one parameter characterize PP

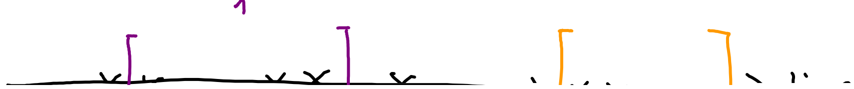
↳ λ (rate) ← same λ that we've seen in $A = \frac{\lambda}{\mu}$

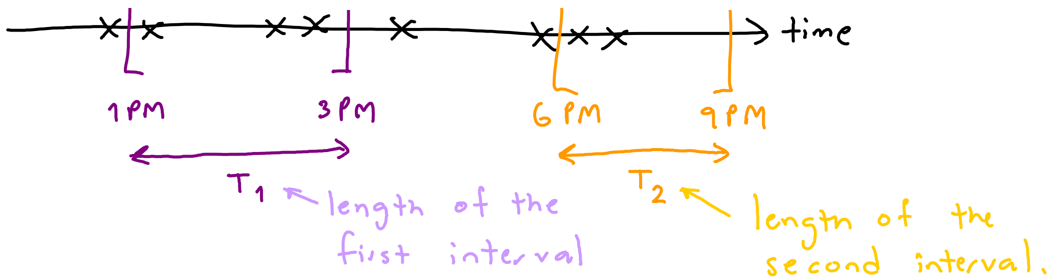
Two types of PP

PP
- homogeneous PP — λ is a constant (we focus on this one.)
- non-homogeneous PP (in-) — λ is a function of time $\lambda(t)$

Unstructured random

$N_1 = 3$ $N_2 = 2$





1^{st} Property : $N_1 \perp\!\!\!\perp N_2$

independent

Extension:

$$\boxed{\begin{array}{l}\text{If you have non-overlapping intervals of time, and} \\ \text{the number of "x" in the } i^{th} \text{ interval } \equiv N_i \leftarrow \text{a random variable (r.v.)} \\ \text{then the random variables } N_1, N_2, N_3, \ldots \text{ are independent.}\end{array}}$$

The "≡" symbol means "is denoted by".

2^{nd} Property : N_i can be $0, 1, 2, 3, \ldots$

discrete → pmf (probability mass function)

$$\left.\begin{array}{l} P[N_i = 0] \\ P[N_i = 1] \\ P[N_i = 2] \\ \vdots \end{array}\right\} \rightarrow P_{N_i}(k) \equiv P[N_i = k]$$

The probability that N_i takes the value k.

mean / expectation / expected value of N_i

3^{rd} Property : $\mathbb{E} N_i = \lambda \times T_i$

length of the i^{th} interval.

For example,

$$\mathbb{E} N_1 = \lambda T_1$$

$$\mathbb{E} N_2 = \lambda T_2$$

Suppose $\lambda = 5$ x's per hour } $\Rightarrow \mathbb{E} N = 10$ x's

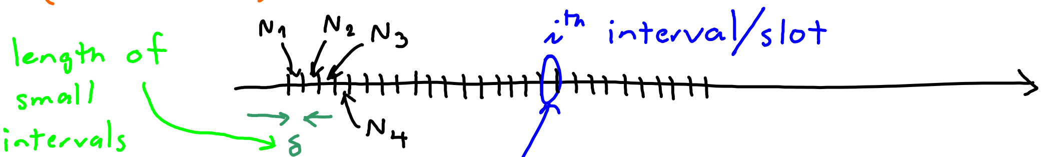
Suppose $\lambda = 5$ x's per hour, $T_1 = 2$ hours $\Big\} \Rightarrow \mathbb{E}N_1 = 10$ x's

In words, ...

For any interval of length T, the expected number of "x"'s in this interval is given by

$$\mathbb{E}N = \lambda T$$

Small Slot Approximation/Analysis
(discrete time)



4^{th} property : When the interval is extremely small, it is unlikely that there will be $\geq$ two "x"'s in the interval.

advantage of considering small intervals

$N_i = 0$ or 1 $\leftarrow$ Bernoulli

What do you remember about Bernoulli r.v.?

From ①, $N_1, N_2, N_3, \ldots$ are independent

③, $\mathbb{E}N_1 = \mathbb{E}N_2 = \mathbb{E}N_3 = \cdots = \lambda\delta$

Lecture 11 (Dec 17)

Implication :

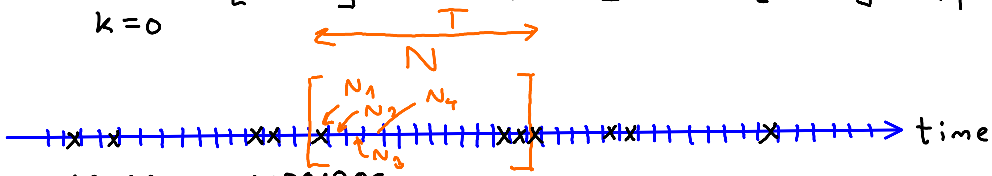
For any interval of "small" length δ, (delta)
the number of "x"'s in this interval can be approximated by a Bernoulli r.v.
The whole PP can be approximated by a sequence of i.i.d. Bernoulli r.v.s with $p_1 = \lambda\delta$.

Bernoulli r.v. X ↗ 0, ↘ 1

$$P[X=0] = p_0 = 1 - p_1$$
$$P[X=1] = p_1$$

$$\mathbb{E}X = \sum_{k}^{1} k \times P[X=k] = 0 \times P[X=0] + 1 \times P[X=1] = p$$

$$\mathbb{E}X = \sum_{k=0}^{1} k \times P[X=k] = 0 \times P[X=0] + 1 \times P[X=1] = p_1$$



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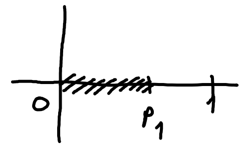


MATLAB

rand$(\underbrace{1, n}_{\text{size}}) < p_1$

or

binornd$(1, p_1, 1, n)$

$$N = N_1 + N_2 + N_3 + N_4 + \cdots + N_n \leftarrow \text{Summation of } n \text{ i.i.d. Bernoulli r.v. with parameter } p_1 \text{ is Binomial}(n, p_1)$$

$$\delta = \frac{T}{n}$$

$$n \to \infty$$

Characteristic Function

$$\varphi_X(u) = \mathbb{E}\, e^{jXu}$$

For Bernoulli r.v.,

$$\varphi_{N_i}(u) = \sum_{k=0}^{1} e^{jku} \times P[X=k]$$

$$= e^{j0u} p_0 + e^{j1u} p_1 = p_0 + p_1 e^{ju}$$

$$p_0 = 1 - \lambda\delta, \quad p_1 = \lambda\delta, \quad \delta = \frac{T}{n}$$

$$= 1 - \lambda\frac{T}{n} + \lambda\frac{T}{n} e^{ju}$$

$$\varphi_N(u) = \varphi_{N_1}(u) \times \varphi_{N_2}(u) \times \varphi_{N_3}(u) \times \cdots \times \varphi_{N_n}(u)$$

$$= \left(\varphi_{N_1}(u)\right)^n = \left(1 - \lambda\frac{T}{n} + \lambda\frac{T}{n} e^{ju}\right)^n$$

Take limit as $n \to \infty$

$$= \left(1 + \frac{1}{n}\left(-\lambda T + \lambda T e^{ju}\right)\right)^n$$

$$= \left(1 + \frac{1}{n} x\right)^n \to e^{+x}$$

$$(-\lambda T + \lambda T e^{ju})$$

$$\rightarrow e^{(-\lambda T + \lambda T e^{j\omega})} \Rightarrow \text{Poisson r.v. } N \quad \mathcal{P}(\lambda T)$$

check this.

$$P[N=k] = e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

$$\varphi_N(\omega) = \sum_{k=0}^{\infty} e^{jk\omega} e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

$$\left(e^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \right)$$

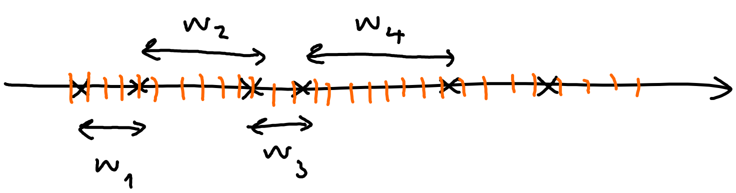
Conclusion

If N is the number of arrivals in an interval of length T,

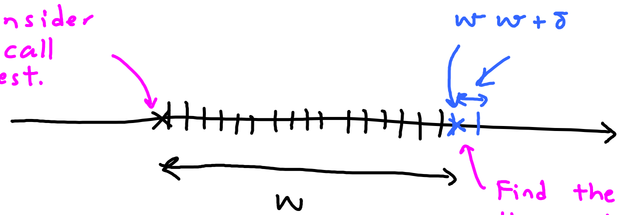
then N is a Poisson r.v. with mean λT.

Lecture 12 (Dec 21)

Back to PP: One more Property of PP



Let's consider this call request.



Find the probability that the next call request happens in this slot.

$$① \; P[W \text{ in the blue slot}] = \int_{w}^{w+\delta} f_W(\tau)\, d\tau \approx f_W(w) \times \delta$$

② how many slot? $\approx \frac{w}{\delta}$ $\quad \frac{w}{\delta} - 1$

$$P[w \text{ in the blue slot}] = (1-\lambda\delta)^{\frac{w}{\delta}-1}(\lambda\delta)$$

These two quantities should be the same.

$$f_W(w) \times \cancel{\delta} = (1-\lambda\delta)^{\frac{w}{\delta}-1} \lambda \cancel{\delta}$$

Let $\delta \to 0$ (make slots small)

$$f_W(w) = \lim_{\delta \to 0} (1-\lambda\delta)^{\frac{w}{\delta}-1} \lambda$$

$$= \lim_{\frac{1}{\delta} \to \infty} \left(\left(1 - \frac{\lambda}{1/\delta}\right)^{\frac{1}{\delta}} \right)^{w} \lambda (1-\lambda\delta)^{-1}$$

$$= e^{-\lambda w} \times \lambda \times 1$$

$$= \lambda e^{-\lambda w} \quad \leftarrow \text{exponential}$$