Name ________________________ ID3 _ _ _

ECS 452: Digital Communication Systems **2019/2**

HW 8 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Consider a convolutional encoder whose trellis diagram is given in Figure 8.1.

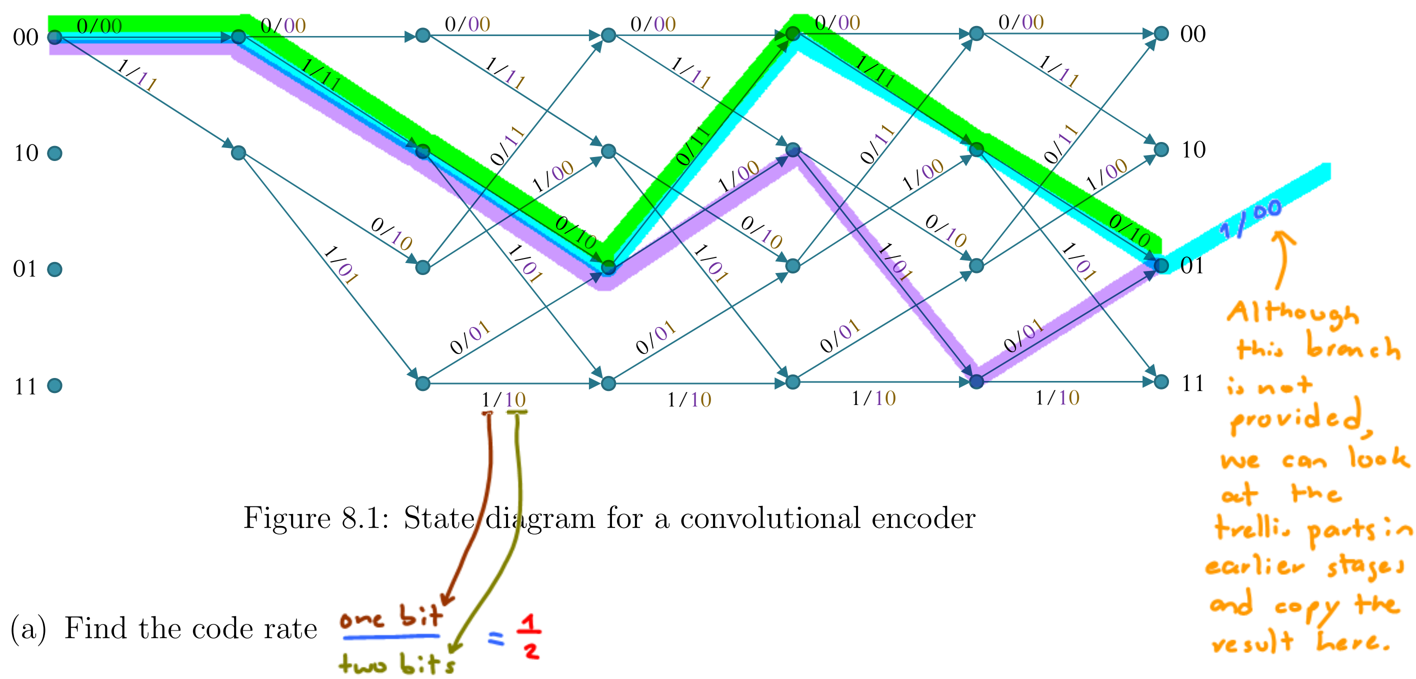


Figure 8.1: State diagram for a convolutional encoder

(a) Find the code rate $\frac{\text{one bit}}{\text{two bits}} = \frac{1}{2}$

(b) Suppose the data bits (message) are $\underline{\mathbf{b}} = [0100101]$. Find the corresponding codeword $\underline{\mathbf{x}}$.

From the blue highligthed path, we have $\underline{x}$ = [00111011111000]

(c) Find the data vector $\underline{\mathbf{b}}$ which gives the codeword $\underline{\mathbf{x}} = [001110111110]$.

From the green highlighted path, we have $\underline{b}$ = [010010]

Alternatively, because the codeword here is the same as the first 12 bits of the codeword in the previous part, we know that the data vector must be the same as the first 6 bits of the data vector from the previous part.

(d) Suppose that we observe $\underline{\mathbf{y}} = [001110000101]$ at the input of the minimum distance decoder. Explain why we can easily find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$ without applying the Viterbi algorithm.

From the purple highligthed path, we see that $\underline{y}$ itself is a codeword.

So, there is a codeword with distance = 0 from $\underline{y}$.

There can not be any codeword with smaller distance from $\underline{y}$ than 0.

So, $\hat{\underline{x}} = \underline{y}$.

From the purple highligthed path, we can also read $\hat{\underline{b}} = [010110]$

(e) Suppose that we observe $\underline{\mathbf{y}} = [010101111110]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$. Show your work on Figure 8.2 below.

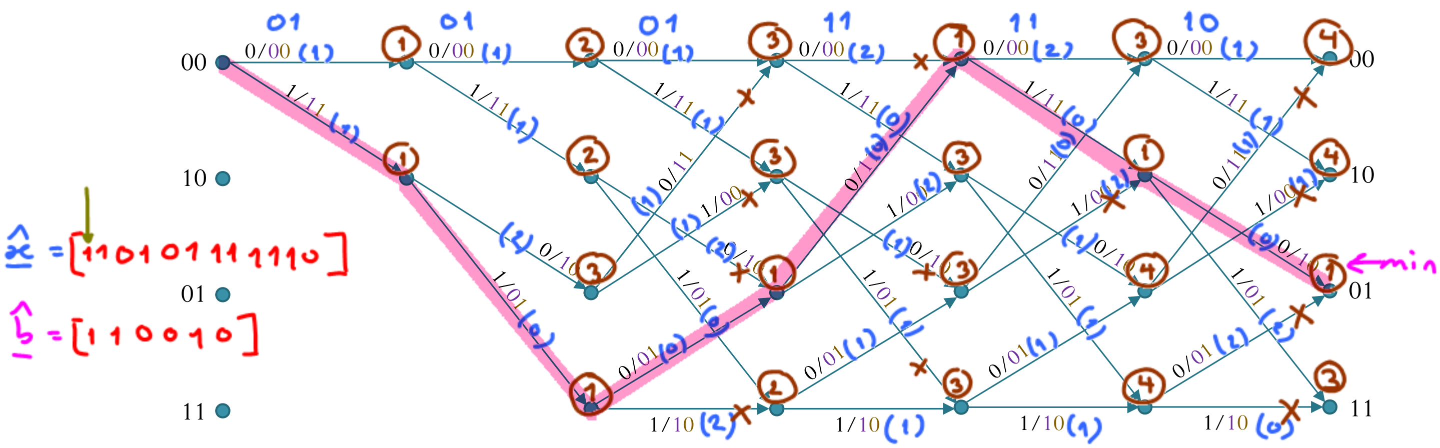


Figure 8.2: State diagram for a convolutional encoder

Make sure that all the running (cumulative) path metric are shown and the discarded branches are indicated at every steps.

Problem 2. Consider four vectors:

$$\mathbf{v}^{(1)} = \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} +1 \\ +1 \\ 0 \\ +1 \\ 0 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} +2 \\ 0 \\ +1 \\ +1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} +3 \\ +1 \\ +1 \\ +2 \\ -1 \end{pmatrix}.$$

Now, consider two vectors:

$$\mathbf{e}^{(1)} = \frac{1}{2}\begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \\ -1 \end{pmatrix}, \text{ and } \mathbf{e}^{(2)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

(a) Show that the two vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are orthonormal.

$$\|\vec{e}^{(1)}\|^2 = \left(\tfrac{1}{2}\right)^2\left(1^2+(-1)^2+1^2+0^2+(-1)^2\right) = \tfrac{1}{4}(1+1+1+0+1) = 1 \Rightarrow \|\vec{e}^{(1)}\| = 1$$

$$\|\vec{e}^{(2)}\|^2 = \left(\tfrac{1}{\sqrt{3}}\right)^2\left(1^2+1^2+0^2+1^2+0^2\right) = \tfrac{1}{3}(3) = 1 \Rightarrow \|\vec{e}^{(2)}\| = 1$$

Both of them have unit length.

orthonormal

$$\langle \vec{e}^{(1)}, \vec{e}^{(2)} \rangle = \tfrac{1}{2} \times \tfrac{1}{\sqrt{3}} \times \left((1)(1) + (-1)(1) + (1)(0) + (0)(1) + (-1)(0)\right)$$

$$= \tfrac{1}{2\sqrt{3}}(1-1+0+0-0) = \tfrac{1}{2\sqrt{3}} \times 0 = 0$$ $\Rightarrow$ They are orthogonal.

(b) Find the corresponding vectors $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, $\mathbf{c}^{(3)}$, and $\mathbf{c}^{(4)}$ that represent $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$, and $\mathbf{v}^{(4)}$ in the new coordinate system defined by vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$.

$$\left.\begin{aligned}\langle \vec{v}^{(1)}, \vec{e}^{(1)} \rangle &= \tfrac{1}{2}(1+1+1+0+1) = \tfrac{4}{2} = 2 \\ \langle \vec{v}^{(1)}, \vec{e}^{(2)} \rangle &= \tfrac{1}{\sqrt{3}}(1-1+0+0+0) = 0\end{aligned}\right\} \Rightarrow \vec{c}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned}\langle \vec{v}^{(2)}, \vec{e}^{(1)} \rangle &= \tfrac{1}{2}(1-1+0+0+0) = 0 \\ \langle \vec{v}^{(2)}, \vec{e}^{(2)} \rangle &= \tfrac{1}{\sqrt{3}}(1+1+0+1+0) = \tfrac{3}{\sqrt{3}} = \sqrt{3}\end{aligned}\right\} \Rightarrow \vec{c}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

$$\left.\begin{aligned}\langle \vec{v}^{(3)}, \vec{e}^{(1)} \rangle &= \tfrac{1}{2}(2+0+1+0+1) = \tfrac{4}{2} = 2 \\ \langle \vec{v}^{(3)}, \vec{e}^{(2)} \rangle &= \tfrac{1}{\sqrt{3}}(2+0+0+1+0) = \tfrac{3}{\sqrt{3}} = \sqrt{3}\end{aligned}\right\} \Rightarrow \vec{c}^{(3)} = \begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix}$$

$$\left.\begin{aligned}\langle \vec{v}^{(4)}, \vec{e}^{(1)} \rangle &= \tfrac{1}{2}(3-1+1+0+1) = \tfrac{4}{2} = 2 \\ \langle \vec{v}^{(4)}, \vec{e}^{(2)} \rangle &= \tfrac{1}{\sqrt{3}}(3+1+0+2+0) = \tfrac{6}{\sqrt{3}} = 2\sqrt{3}\end{aligned}\right\} \Rightarrow \vec{c}^{(4)} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$$

Problem 3. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 8.4. Note that V and T_b are some positive constants. Your answers should be given in terms of them.

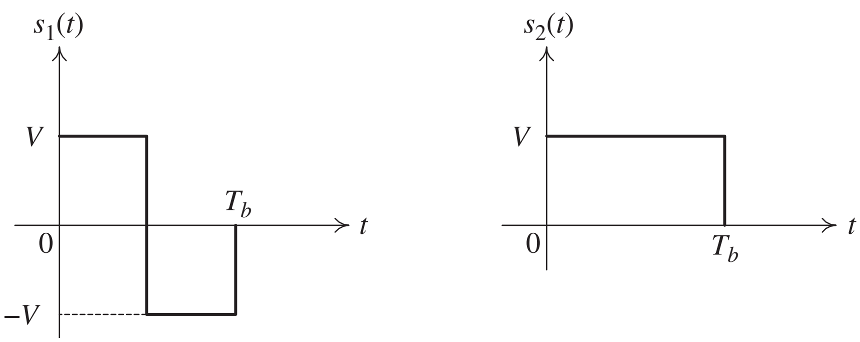


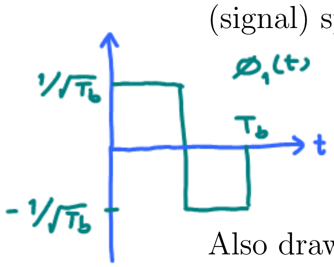
Figure 8.4: Signal set for Problem 3

(a) Find the energy in each signal.

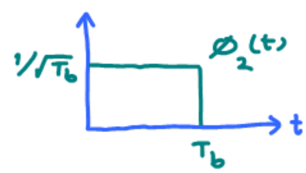
$$E_1 = E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b. \quad \equiv E_s$$

$$E_2 = E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

(b) Find the two vectors that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space with respect to the following orthonormal vectors:



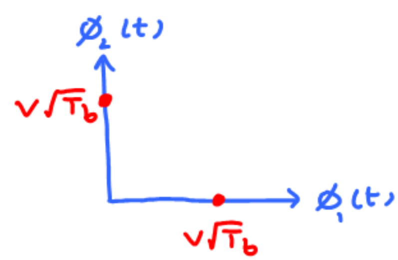
$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 \le t \le \frac{T_b}{2}, \\ -\frac{1}{\sqrt{T_b}}, & \frac{T_b}{2} < t < T_b, \\ 0, & \text{otherwise} \end{cases} \qquad \phi_2(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 \le t \le T_b, \\ 0, & \text{otherwise} \end{cases}$$



Also draw the corresponding constellation.

$$\left.\begin{aligned} \langle s_1(t), \phi_1(t)\rangle &= \int_{-\infty}^{\infty} s_1(t)\phi_1(t)\,dt = (V)\left(\tfrac{1}{\sqrt{T_b}}\right)\left(\tfrac{T_b}{2}\right) + (-V)\left(-\tfrac{1}{\sqrt{T_b}}\right)\left(\tfrac{T_b}{2}\right) = V\sqrt{T_b} \\ \langle s_1(t), \phi_2(t)\rangle &= \left(\tfrac{1}{\sqrt{T_b}}\right)\left(V\tfrac{T_b}{2} + (-V)\tfrac{T_b}{2}\right) = 0 \end{aligned}\right\} \vec{s}^{(1)} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \langle s_2(t), \phi_1(t)\rangle &= V\left(\left(\tfrac{1}{\sqrt{T_b}}\right)\tfrac{T_b}{2} + \left(-\tfrac{1}{\sqrt{T_b}}\right)\tfrac{T_b}{2}\right) = 0 \\ \langle s_2(t), \phi_2(t)\rangle &= V\left(\tfrac{1}{\sqrt{T_b}}\right)(T_b) = V\sqrt{T_b} \end{aligned}\right\} \vec{s}^{(2)} = \begin{pmatrix} 0 \\ V\sqrt{T_b} \end{pmatrix}$$



Problem 4. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 8.5. Note that V, α and T_b are some positive constants.

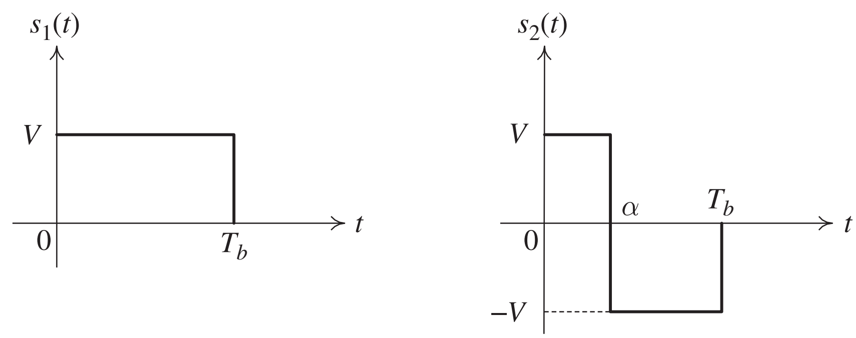


Figure 8.5: Signal set for Problem 4

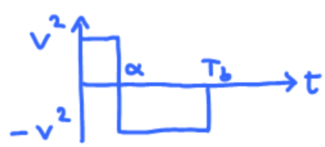
(a) Find the energy in each signal.

$$E_1 = E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$$\equiv E_s$$

$$E_2 = E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

(b) Find $\langle s_1(t), s_2(t)\rangle$.

$= \int$ 

$dt = \alpha V^2 - (T_b - \alpha) V^2 = 2\alpha V^2 - T_b V^2$

(c) Consider two orthonormal vectors:

$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 < t < T_b, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \phi_2(t) = \frac{1}{\sqrt{\alpha\left(1 - \frac{\alpha}{T_b}\right)}} \times \begin{cases} 1 - \frac{\alpha}{T_b} & 0 < t \le \alpha, \\ -\frac{\alpha}{T_b}, & \alpha < t < T_b, \\ 0, & \text{otherwise} \end{cases}$$

Let's refer to this as β

(i) Check that they are orthonormal.

$$\|\phi_1\|^2 = \int_{-\infty}^{\infty} (\phi_1(t))^2 dt = \left(\frac{1}{\sqrt{T_b}}\right)^2 \times T_b = 1$$

$$\|\phi_2\|^2 = \int_{-\infty}^{\infty} (\phi_2(t))^2 dt = \frac{1}{\alpha(1-\frac{\alpha}{T_b})}\left(\left(1-\frac{\alpha}{T_b}\right)^2 \alpha + \left(+\frac{\alpha}{T_b}\right)^2 (T_b-\alpha) \right) = 1$$

$$\langle \phi_1(t), \phi_2(t) \rangle = \int_{-\infty}^{\infty} \phi_1(t)\phi_2(t) dt = \frac{1}{\sqrt{T_b}}\beta\left(\left(1-\frac{\alpha}{T_b}\right)\alpha + \left(-\frac{\alpha}{T_b}\right)(T_b-\alpha) \right) = 0$$

(ii) Find the two vectors $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space with respect to the given orthonormal vectors.

$$\langle s_1(t), \phi_1(t) \rangle = \int_{-\infty}^{\infty} s_1(t)\phi_1(t) dt = \int_0^{T_b} \frac{V}{\sqrt{T_b}} dt = \frac{V}{\sqrt{T_b}} T_b = V\sqrt{T_b}$$

$$\langle s_1(t), \phi_2(t) \rangle = \int_{-\infty}^{\infty} s_1(t)\phi_2(t) dt = \beta V \left(\left(1-\frac{\alpha}{T_b}\right)\alpha + \left(-\frac{\alpha}{T_b}\right)(T_b-\alpha) \right) = 0$$

$$\vec{s}^{(1)} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}.$$

$$\langle s_2(t), \phi_1(t) \rangle = \int_{-\infty}^{\infty} s_2(t)\phi_1(t) dt = \frac{1}{\sqrt{T_b}}\left(V\alpha + (-V)(T_b-\alpha) \right) = \frac{V}{\sqrt{T_b}}(2\alpha - T_b)$$

$$= V\sqrt{T_b}\left(2\frac{\alpha}{T_b} - 1\right)$$

$$\langle s_2(t), \phi_2(t) \rangle = \int_{-\infty}^{\infty} s_2(t)\phi_2(t) dt = \beta\left(V\left(1-\frac{\alpha}{T_b}\right)(\alpha) + (+V)\left(+\frac{\alpha}{T_b}\right)(T_b-\alpha) \right)$$

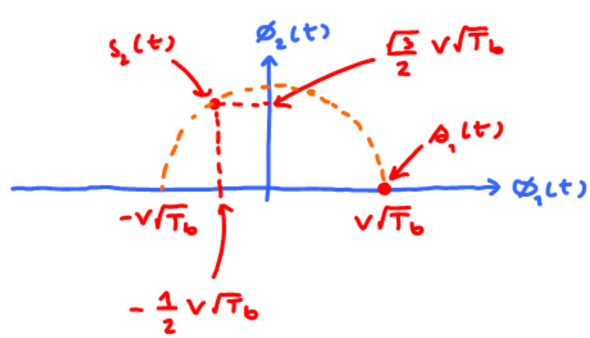
$$= 2\beta V\left(1-\frac{\alpha}{T_b}\right)\alpha = V\sqrt{T_b}\, 2\sqrt{\frac{\alpha}{T_b}\left(1-\frac{\alpha}{T_b}\right)}$$

$$\vec{s}^{(2)} = V\sqrt{T_b} \times \begin{pmatrix} 2\frac{\alpha}{T_b} - 1 \\ 2\sqrt{\frac{\alpha}{T_b}\left(1-\frac{\alpha}{T_b}\right)} \end{pmatrix} = \sqrt{E_s}\begin{pmatrix} 2r-1 \\ 2\sqrt{r(1-r)} \end{pmatrix}$$

where $r = \frac{\alpha}{T_b}$.

(iii) Draw the corresponding constellation when $\alpha = \frac{T_b}{4}$.

When $\alpha = \frac{T_b}{4}$, $s_2(t) = 2V\sqrt{\frac{T_b}{4}\left(\frac{3}{4}\right)}\,\phi_2(t) - \left(1-\frac{1}{2}\right)V\sqrt{T_b}\,\phi_1(t) = \frac{\sqrt{3}}{2}V\sqrt{T_b}\,\phi_2(t) - \frac{1}{2}V\sqrt{T_b}\,\phi_1(t)$

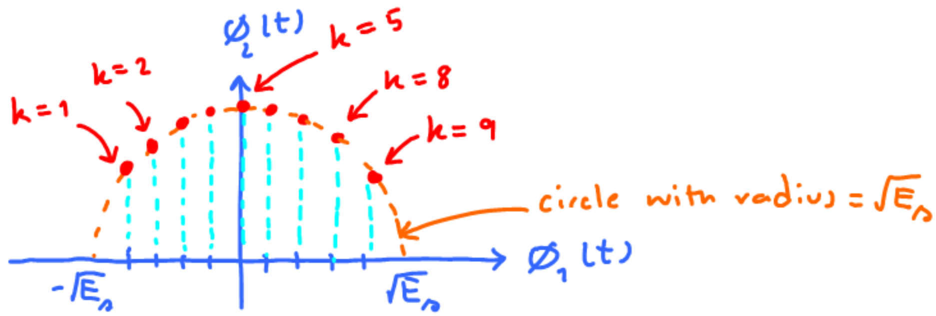


(iv) Draw $\mathbf{s}^{(2)}$ when $\alpha = \frac{k}{10}T_b$ where $k = 1, 2, \dots, 9$.

Note that

$$(2r-1)^2 + \left(2\sqrt{r(1-r)}\right)^2 = 4r^2 - 4r + 1 + 4r - 4r^2 = 1.$$

$$r = \frac{\alpha}{T_b} = \frac{k}{10}$$



Extra Question

Here is an optional question for those who want more practice.

Problem 5. Consider a convolutional code generated by the encoder shown in Figure 8.3. Suppose that we observe $\underline{\mathbf{y}} = [110111000110]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$. Caution: The trellis diagram is not the same as the one used in Problem 1.

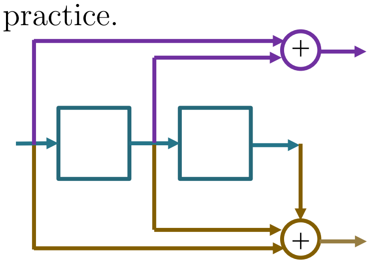


Figure 8.3: Encoder for Problem 5

Solution: Observe that the circuit diagram is exactly the same as the one used in Exercise 15. We have already found its state diagram; this is shown in Figure 8.6a. From the state diagram, we can then create the code trellis shown in Figure 8.6b.

For Viterbi decoding, the trellis diagram is shown in Figure 8.7. (Don't forget that the trellis diagram always starts with the all-zero state.)

Tracing back the trellis diagram, we get $\boxed{\hat{\underline{\mathbf{b}}} = [101110]}$ and $\boxed{\hat{\underline{\mathbf{x}}} = [111110000110]}$.

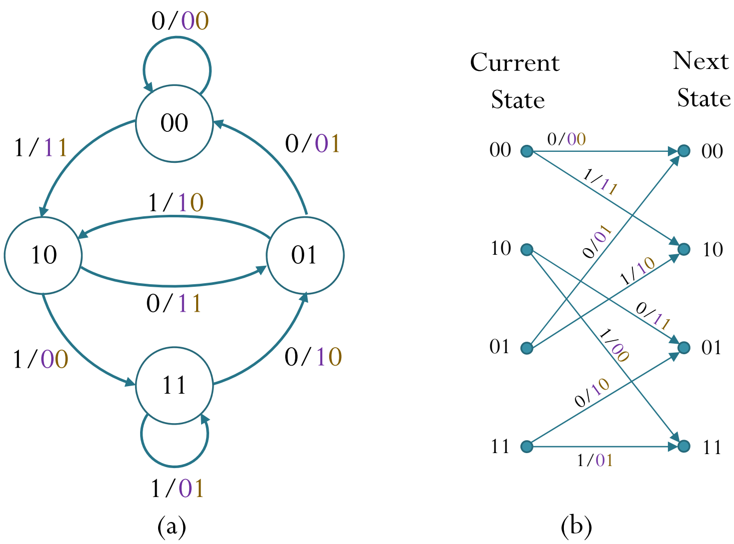


Figure 8.6: (a) State diagram and (b) Code Trellis for Problem 5

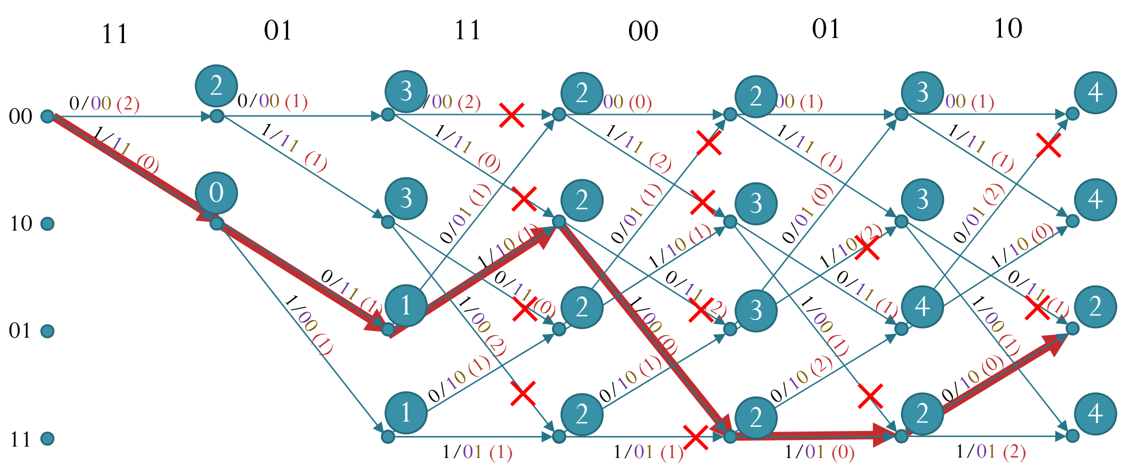


Figure 8.7: Trellis diagram for Problem 5.

Problem 6. Suppose $s_1(t) = \mathrm{sinc}(5t)$ and $s_2(t) = \mathrm{sinc}(7t)$. Note that in this class, we define $\mathrm{sinc}(x) = \frac{\sin x}{x}$. Find

(a) E_{s_1}, $= \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_{-\infty}^{\infty} |S_1(f)|^2 df$

(b) E_{s_2}, and $= \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_{-\infty}^{\infty} |S_2(f)|^2 df$

(c) $\langle s_1(t), s_2(t)\rangle$. $= \int_{-\infty}^{\infty} s_1(t) s_2^*(t)\, dt = \int_{-\infty}^{\infty} S_1(f) S_2^*(f)\, df.$

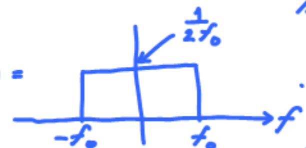
This is not needed here because the waveforms are real-valued.

Eventhough the waveforms are real-valued in the time domian, their Fourier transforms may not be real-valued. Therefore, the complex-conjugation is still needed here.

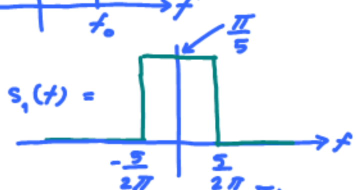
Hint: Use Parseval's theorem to evaluate the above quantities in the frequency domain. (Review: See 2.43 on p. 22 and Example 2.44 on p. 23 of ECS332 2019 lecture notes and Problem 6 in ECS332 2019 HW4.)

Our first step is to find the Fourier transforms $S_1(f)$ and $S_2(f)$ of the waveforms $s_1(t)$ and $s_2(t)$.

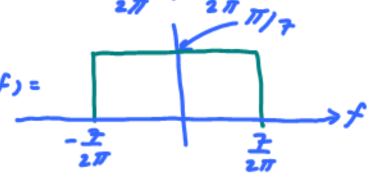
For $g(t) = \mathrm{sinc}(2\pi f_0 t)$, we have seen that $G(f) =$



Therefore, for $s_1(t) = \mathrm{sinc}(5t)$, we have $f_0 = \frac{5}{2\pi}$ and $S_1(f) =$



Similarly, for $s_2(t) = \mathrm{sinc}(7t)$, we have $f_0 = \frac{7}{2\pi}$ and $S_2(f) =$



(a) $E_{s_1} = \int_{-\infty}^{\infty} |S_1(f)|^2 df = \left(\frac{\pi}{5}\right)^2 \times \left(2 \times \frac{5}{2\pi}\right) = \frac{\pi}{5}$

(b) $E_{s_2} = \int_{-\infty}^{\infty} |S_2(f)|^2 df = \left(\frac{\pi}{7}\right)^2 \times \left(2 \times \frac{7}{2\pi}\right) = \frac{\pi}{7}$

(c) $\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} S_1(f) S_2^*(f)\, df = \frac{\pi}{5} \times \frac{\pi}{7} \times 2 \times \frac{5}{2\pi} = \frac{\pi}{7}$

It turns out that $S_2(f)$ is real-valued here. So, it is safe to ignore the complex-conjugation.

Problem 7. Prove the following facts with the help of Fourier transform.
(Hint: inner product in the frequency domain, Parseval's theorem)

(a) The energy of $p(t) = g(t)\cos(2\pi f_c t + \phi)$ is $E_g/2$.

(b) $g(t)\cos(2\pi f_c t)$ and $-g(t)\sin(2\pi f_c t)$ are orthogonal.

Is there any condition(s) on $g(t)$ that we need to assume?

(a)

$$p(t) = g(t)\cos(2\pi f_c t + \phi) = \frac{g(t)}{2}\left(e^{j(2\pi f_c t + \phi)} + e^{-j(2\pi f_c t+\phi)}\right)$$

$$\cos A = \frac{e^{jA} + e^{-jA}}{2}$$

$$= \frac{g(t)}{2} e^{j\phi} e^{j2\pi f_c t} + \frac{g(t)}{2} e^{-j\phi} e^{-j2\pi f_c t}$$

$$\Big\downarrow \mathcal{F}$$

$$P(f) = \frac{e^{j\phi}}{2} G(f-f_c) + \frac{e^{-j\phi}}{2} G(f+f_c)$$

$$E_p = \langle p(t), p(t)\rangle = \langle P(f), P(f)\rangle = \int_{-\infty}^{\infty} P(f)\, P^*(f)\, df$$

$$= \int_{-\infty}^{\infty} \Big(\underbrace{\frac{e^{j\phi}}{2} G(f-f_c)}_{A} + \underbrace{\frac{e^{-j\phi}}{2} G(f+f_c)}_{B}\Big)\Big(\underbrace{\frac{e^{-j\phi}}{2} G^*(f-f_c)}_{C} + \underbrace{\frac{e^{j\phi}}{2} G^*(f+f_c)}_{D}\Big)\, df$$

$$= \int_{-\infty}^{\infty} (A+B)(C+D)\, df = \int_{-\infty}^{\infty} AC\, df + \int_{-\infty}^{\infty} AD\, df + \int_{-\infty}^{\infty} BC\, df + \int_{-\infty}^{\infty} BD\, df$$

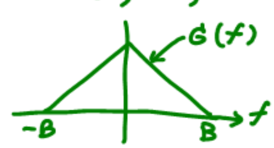
Now, note that

$$AC = \frac{1}{4}|G(f-f_c)|^2$$

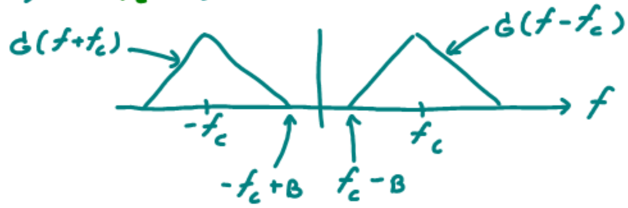
$$BD = \frac{1}{4}|G(f+f_c)|^2$$

suppose we assume that $G(f)$ is band-limited to $\pm B$, i.e.,

$$G(f) = 0 \quad \text{when} \quad |f| > B.$$



Then, if $f_c > B$,



the non-zero parts of $G(f-f_c)$ and $G(f+f_c)$ do not overlap.

Therefore, $\left.\begin{array}{l} BC \equiv 0 \\ AD \equiv 0 \end{array}\right\}$ across all f.

So, $E_p = \int_{-\infty}^{\infty} \frac{1}{4}|G(f-f_c)|^2 df + 0 + 0 + \int_{-\infty}^{\infty} \frac{1}{4}|G(f+f_c)|^2 df$

$\mu = f - f_c$ $\qquad$ $\mu = f + f_c$ $\qquad$ (change of variables)

$$= \int_{-\infty}^{\infty} \frac{1}{4}|G(\mu)|^2 d\mu + \int_{-\infty}^{\infty} \frac{1}{4}|G(\mu)|^2 d\mu$$

$$= \frac{1}{4}E_g + \frac{1}{4}E_g = \frac{E_g}{2}.$$

This formula is valid when $G(f) = 0$ for $|f| \geq f_c$.

Note that there are other situations that can make $E_p = \frac{1}{2}E_g$ as well.
For example, in Problem 5c, we have signals of the form

$$p(t) = g(t)\cos\left(2\pi \frac{m}{T_0} t\right) \quad \text{where} \quad g(t) = 1_{[0,T_0]}(t).$$

There, we show (by direct integration) that $E_p = \frac{T_0}{2}$.
Note that $E_g = \int_{-\infty}^{\infty} g^2(t)\,dt = \int_0^{T_0} 1^2\,dt = T_0$. Therefore, $E_p = \frac{1}{2}E_g$.
However, there, $g(t)$ is time-limited and hence cannot be band-limited.

(b)

$$x(t) = g(t)\cos(2\pi f_c t) \xrightarrow{\mathcal{F}} X(f) = \frac{1}{2}\Big(\overbrace{G(f-f_c)}^{A} + \overbrace{G(f+f_c)}^{B}\Big)$$

$$y(t) = -g(t)\underbrace{\sin(2\pi f_c t)}_{=\frac{1}{2j}\left(e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right)} \xrightarrow{\mathcal{F}} Y^*(f) = +\frac{1}{2j}\Big(\underbrace{G^*(f-f_c)}_{C} - \underbrace{G^*(f+f_c)}_{D}\Big)$$

$$\langle x,y\rangle = \langle X,Y\rangle = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df = \frac{1}{4j}\int_{-\infty}^{\infty} \underbrace{(A+B)(C-D)}_{=AC-AD+BC-BD}\,df$$

$$= \frac{1}{4j}\int_{-\infty}^{\infty} |G(f-\cancel{f_c})|^2 df - \frac{1}{4j}\int_{-\infty}^{\infty} |G(f+\cancel{f_c})|^2 df + 0 + 0$$

$$= 0$$

$G(f-f_c)$ and $G(f+f_c)$ do not overlap.

Assumption: $G(f)$ is bandlimited

$G(f) = 0$ for $f > f_c$, $f < -f_c$