

ECS 452: In-Class Exercise # 1 Sol

Instructions

1. Separate into groups of no more than three students each.
2. [ENRE] Explanation is not required for this exercise.
3. **Do not panic.**

Date: 17 / 1 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider two codes (for source coding) below. The left column is for Code A. The right column is for Code B. The first row defines these codes via their codebooks.

<p>Codebook for Code A</p> <table border="1" style="margin: 10px auto; border-collapse: collapse; text-align: center;"> <tr> <td>x</td> <td>a</td> <td>e</td> <td>ℓ</td> <td>n</td> <td>r</td> </tr> <tr> <td>$c(x)$</td> <td>1</td> <td>00</td> <td>010</td> <td>0110</td> <td>0111</td> </tr> </table>	x	a	e	ℓ	n	r	$c(x)$	1	00	010	0110	0111	<p>Codebook for Code B</p> <table border="1" style="margin: 10px auto; border-collapse: collapse; text-align: center;"> <tr> <td>x</td> <td>a</td> <td>e</td> <td>ℓ</td> <td>n</td> <td>r</td> </tr> <tr> <td>$c(x)$</td> <td>000</td> <td>100</td> <td>10</td> <td>01</td> <td>11</td> </tr> </table>	x	a	e	ℓ	n	r	$c(x)$	000	100	10	01	11		
x	a	e	ℓ	n	r																						
$c(x)$	1	00	010	0110	0111																						
x	a	e	ℓ	n	r																						
$c(x)$	000	100	10	01	11																						
<p>The source alphabet for Code A is $\{a, e, \ell, n, r\}$. A codebook shows how each source symbol is mapped into a codeword. The set of all possible source symbols is called the source alphabet. Don't forget that it is a set.</p>	<p>The source alphabet for Code B is $\{a, e, \ell, n, r\}$. The first row in the codebook is the same as the one in Code A. Therefore, the possible source symbols are the same as Code A.</p>																										
<p>The code alphabet for Code A is $\{0,1\}$. Note that 1, 00, 010, 0110, 0111 are codewords. These codewords are constructed from code symbols: 0 or 1. So, the code alphabet, which is the set of all possible code symbols, is $\{0,1\}$.</p>	<p>The code alphabet for Code B is $\{0,1\}$. Although the codewords for Code B are different from Code A, they are still constructed from code symbols: 0 or 1. So, the code alphabet is still $\{0,1\}$.</p>																										
<p>Use code A to encode the source string "real"</p> <table style="margin: 10px auto; border-collapse: collapse; text-align: center;"> <tr> <td>r</td><td>e</td><td>a</td><td>ℓ</td> </tr> <tr> <td style="border-right: 1px solid black;">0111</td> <td style="border-right: 1px solid black;">00</td> <td style="border-right: 1px solid black;">010</td> <td>1010</td> </tr> </table>	r	e	a	ℓ	0111	00	010	1010	<p>Use code B to encode the source string "real"</p> <table style="margin: 10px auto; border-collapse: collapse; text-align: center;"> <tr> <td>r</td><td>e</td><td>a</td><td>ℓ</td> </tr> <tr> <td style="border-right: 1px solid black;">111</td> <td style="border-right: 1px solid black;">000</td> <td style="border-right: 1px solid black;">100</td> <td>10</td> </tr> </table>	r	e	a	ℓ	111	000	100	10										
r	e	a	ℓ																								
0111	00	010	1010																								
r	e	a	ℓ																								
111	000	100	10																								
<p>Is Code A nonsingular? By definition, a nonsingular code is a code in which every source symbol in the source alphabet is mapped to a unique codeword. Here, the codewords in the codebook are all different. So, yes, Code A is nonsingular.</p>	<p>Is Code B nonsingular? The codewords in the codebook are all different. So, yes, Code B is nonsingular.</p>																										
<p>The string 01000101110110 comes from encoding a source string by Code A. Decode it.</p> <table style="margin: 10px auto; border-collapse: collapse; text-align: center;"> <tr> <td style="border-right: 1px solid black;">010</td> <td style="border-right: 1px solid black;">00</td> <td style="border-right: 1px solid black;">1</td> <td style="border-right: 1px solid black;">0111</td> <td>0110</td> </tr> <tr> <td style="border-right: 1px solid black;">ℓ</td> <td style="border-right: 1px solid black;">e</td> <td style="border-right: 1px solid black;">a</td> <td style="border-right: 1px solid black;">r</td> <td>n</td> </tr> </table> <p>The decoded string is learn.</p>	010	00	1	0111	0110	ℓ	e	a	r	n	<p>The string 0001110001000 comes from encoding a source string by Code B. Decode it.</p> <p>First, we try to decode the beginning part of the encoded string:</p> <table style="margin: 10px auto; border-collapse: collapse; text-align: center;"> <tr> <td style="border-right: 1px solid black;">000</td> <td style="border-right: 1px solid black;">11</td> <td>10001000</td> </tr> <tr> <td style="border-right: 1px solid black;">a</td> <td style="border-right: 1px solid black;">r</td> <td>?</td> </tr> </table> <p>The next step is tricky because the next codeword could be 10 (ℓ) or 100 (e). However, if 10 (ℓ) is used here, no codeword can start the remaining string 001000. Therefore, there is only one interpretation:</p> <table style="margin: 10px auto; border-collapse: collapse; text-align: center;"> <tr> <td style="border-right: 1px solid black;">000</td> <td style="border-right: 1px solid black;">11</td> <td style="border-right: 1px solid black;">100</td> <td style="border-right: 1px solid black;">01</td> <td>000</td> </tr> <tr> <td style="border-right: 1px solid black;">a</td> <td style="border-right: 1px solid black;">r</td> <td style="border-right: 1px solid black;">e</td> <td style="border-right: 1px solid black;">n</td> <td>a</td> </tr> </table> <p>The decoded string is arena.</p>	000	11	10001000	a	r	?	000	11	100	01	000	a	r	e	n	a
010	00	1	0111	0110																							
ℓ	e	a	r	n																							
000	11	10001000																									
a	r	?																									
000	11	100	01	000																							
a	r	e	n	a																							

ECS 452: In-Class Exercise # 2 Sol

Instructions

- Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
- Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
- Do not panic.**

Date: 21 / 1 / 2020

Name

ID (last 3 digits)

- Consider a DMS whose source alphabet is $\{a, e, \ell, n, r\}$.

The probabilities for these five symbols are shown in the table below:

x	a	e	ℓ	n	r
$p(x)$	0.1	0.2	0.2	0.2	0.3

Consider two codes (for source coding) below.

The left column is for Code A. The right column is for Code B.

The first row defines these codes via their codebooks.

Codebook for Code A

x	a	e	ℓ	n	r
$c(x)$	1	00	010	0110	0111
$\ell(x)$	1	2	3	4	4
$p(x)$	0.1	0.2	0.2	0.2	0.3

Codebook for Code B

x	a	e	ℓ	n	r
$c(x)$	000	100	10	01	11
$\ell(x)$	3	3	2	2	2
$p(x)$	0.1	0.2	0.2	0.2	0.3

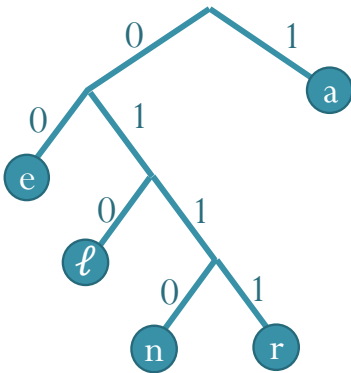
Suppose the DMS above is encoded by Code A.
Find the expected codeword length.

$$\begin{aligned} \mathbb{E}[\ell(X)] &= \sum_x \ell(x)p(x) \\ &= (1 \times 0.1) + ((2 + 3 + 4) \times 0.2) + (4 \times 0.3) \\ &= 0.1 + 1.8 + 1.2 \\ &= 3.1 \text{ [bits per source symbol]} \end{aligned}$$

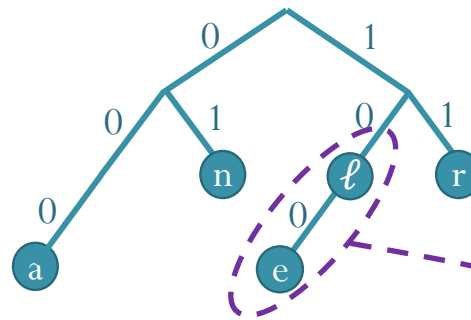
Suppose the DMS above is encoded by Code B.
Find the expected codeword length.

$$\begin{aligned} \mathbb{E}[\ell(X)] &= \sum_x \ell(x)p(x) \\ &= (3 \times (0.1 + 0.2)) + (2 \times (0.2 + 0.2 + 0.3)) \\ &= 0.9 + 1.4 \\ &= 2.3 \text{ [bits per source symbol]} \end{aligned}$$

Draw the code tree for Code A.



Draw the code tree for Code B.



Is Code A prefix-free?

Method 1: We check all pairs of codewords and see whether, in each pair, a codeword is a prefix of another codeword. Here, **yes**, the code is prefix-free because no codeword is a prefix of another codeword.

Method 2: We look at the code tree. Code A is a prefix-free code because all of its codewords are leaves (end nodes).

Is Code B prefix-free?

Method 1: We check all pairs of codewords and see whether, in each pair, a codeword is a prefix of another codeword. Here, **no**, the code is not prefix-free because the codeword 10 is a prefix of the codeword 100.

Method 2: We look at the code tree. Code B is not a prefix-free code because there is an active branch leaving the codeword for ℓ .

ECS 452: In-Class Exercise # 3

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 24 / 1 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider a random variable X which has five possible values. Their probabilities are shown in the table below.

x	$p_X(x)$	$c(x)$	$\ell(x)$	$\log_2 \frac{1}{p_X(x)}$
a	0.2	10	2	2.3219
e	0.1	111	3	3.3219
ℓ	0.1	110	3	3.3219
n	0.3	00	2	1.7370
r	0.3	01	2	1.7370

- a. Find a binary Huffman code (without extension) for this random variable. Put the values of the codewords and the codeword lengths in the table above. Note that the codeword for the source symbol “ℓ” is required to be 110. (So, you do not have complete freedom on assigning 0 and 1 to the edges.)

Step 1: We start with {0.2,0.1,0.1,0.3,0.3}.

We first combine the two smallest probabilities which are 0.1 and 0.1.

Now, we have {0.2,0.2,0.3,0.3}.

Step 2: We combine the two smallest probabilities which are 0.2 and 0.2. This gives {0.4,0.3,0.3}

Step 3: We combine the two smallest probabilities which are 0.3 and 0.3. This gives {0.4,0.6}

Step 4: There are only two probabilities left. So, we combine them. It’s reassuring to have the final sum being 1.

Step 5: The codeword for ℓ must be 110. So, we assign the 0 and 1 to the edges such that this is the corresponding codeword. This also gives the values on the paring edges.

Step 6: The remaining edges have some flexibility. Here we choose to assign 0 to the upper edge and 1 to the lower edge.

- b. Find the expected codeword length (per source symbol) when your Huffman coding in part (a) is used (without extension).

$$\begin{aligned} \mathbb{E}[\ell(X)] &= \sum_x \ell(x)p(x) = (2 \times (0.2 + 0.3 + 0.3)) + (3 \times (0.1 + 0.1)) \\ &= (2 \times 0.8) + (3 \times 0.2) = 1.6 + 0.6 = 2.2 \text{ [bits per source symbol]} \end{aligned}$$

c. Calculate $\sum_x \left(p_X(x) \left(\log_2 \frac{1}{p_X(x)} \right) \right)$

(Note that $\log_2 c = \frac{\log_{10} c}{\log_{10} 2} = \frac{\ln c}{\ln 2}$.)

$$\begin{aligned} &\approx (0.2 \times 2.3219) + (2 \times 0.1 \times 3.3219) + (2 \times 0.3 \times 1.7370) \\ &= 0.4644 + 0.6644 + 1.0422 = 2.1710 \text{ [bits per source symbol]} \end{aligned}$$

ECS 452: In-Class Exercise # 4 Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 28 / 1 / 2020			
Name			ID <small>(last 3 digits)</small>

1. A discrete memoryless source emits three possible messages Yes, No, and OK with probabilities 0.1, 0.1, and 0.8, respectively.
 - a. Find the expected codeword length when a binary Huffman code is constructed without extension.

x	$p(x)$		$\ell(x)$
Yes	0.1	0.2	2
No	0.1		2
OK	0.8	1	1

$$\begin{aligned}
 \mathbb{E}[\ell(X)] &= \sum_x \ell(x)p(x) \\
 &= 2 \times 0.1 \times 2 + 1 \times 0.8 \\
 &= 0.4 + 0.8 \\
 &= \mathbf{1.2 \text{ [bits per source symbol]}}
 \end{aligned}$$

Remark: The problem does not ask us to find the codewords. Only the codeword lengths are needed. Once the tree is formed, we can read the codeword lengths directly.

- b. Find the codeword lengths when binary Huffman code with second-order extension is used to encode this source. Put the values of the corresponding probabilities and the codeword lengths in the table below. (Note that, for brevity, we use Y,N,K to represent Yes, No, and OK, respectively.)

x_1x_2	$p_{X_1, X_2}(x_1, x_2)$		$\ell(x_1, x_2)$
YY	$0.1 \times 0.1 = 0.01$		6
YN	$0.1 \times 0.1 = 0.01$		6
YK	$0.1 \times 0.8 = 0.08$		4
NY	$0.1 \times 0.1 = 0.01$		6
NN	$0.1 \times 0.1 = 0.01$		6
NK	$0.1 \times 0.8 = 0.08$		3
KY	$0.8 \times 0.1 = 0.08$		3
KN	$0.8 \times 0.1 = 0.08$		3
KK	$0.8 \times 0.8 = 0.64$		1

- c. Find L_2 .
(This is the expected codeword length **per source symbol** of the Huffman binary code for the second-order extension of this source.)

$$\begin{aligned}
 \mathbb{E}[\ell(X_1, X_2)] &= \sum_{(x_1, x_2)} \ell(x_1, x_2)p_{X_1, X_2}(x_1, x_2) = 4 \times 6 \times 0.01 + 4 \times 0.08 + 3 \times 3 \times 0.08 + 1 \times 0.64 \\
 &= 0.24 + 0.32 + 0.72 + 0.64 = 1.92 \text{ [bits per two source symbols]} \\
 L_2 &= \frac{1.92}{2} = \mathbf{0.96 \text{ [bits per source symbol]}}
 \end{aligned}$$

ECS 452: In-Class Exercise # 5 Sol

Instructions

- Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
- Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
- Do not panic.**

Date: 31 / 1 / 2020			
Name			ID (last 3 digits)
Prapun			5 5 5

1. In each part below, we consider a random variable X which has five possible values. The probability for each possible value is listed in the provided table. Calculate the corresponding entropy value.

a.

x	a	e	ℓ	n	r
$p(x)$	0.25	0.25	0.125	0.25	0.125

$$\begin{aligned}
 H(X) &= - \sum_x p(x) \log_2 p(x) = -(3 \times 0.25 \times \log_2 0.25 + 2 \times 0.125 \times \log_2 0.125) \\
 &= - \left(3 \times \frac{1}{4} \times \log_2 \frac{1}{4} + 2 \times \frac{1}{8} \times \log_2 \frac{1}{8} \right) = 3 \times \frac{1}{4} \times 2 + 2 \times \frac{1}{8} \times 3 = \frac{6}{4} + \frac{3}{4} = \frac{9}{4} = 2.25 \text{ [bits]}
 \end{aligned}$$

b.

x	a	e	ℓ	n	r
$p(x)$	0.1	0.2	0.2	0.2	0.3

$$\begin{aligned}
 H(X) &= - \sum_x p(x) \log_2 p(x) = -(0.1 \times \log_2 0.1 + 3 \times 0.2 \times \log_2 0.2 + 0.3 \times \log_2 0.3) \\
 &\approx 0.1 \times 3.3219 + 3 \times 0.2 \times 2.3219 + 0.3 \times 1.7370 \approx 0.3322 + 3 \times 0.4644 + 0.5211 \\
 &\approx 2.2464 \text{ [bits]}
 \end{aligned}$$

2. [ENRPr] In each row of the following table, compare the quantity in the first column with the one in the third column by writing “>”, “=”, or “<” in the second column. Watch out for approximation error.

(a)	$H(\underline{\mathbf{p}}) \approx 0.9710$ when $\underline{\mathbf{p}} = [0.4, 0.6]$	>	$H(\underline{\mathbf{p}}) \approx 0.8813$ when $\underline{\mathbf{p}} = [0.3, 0.7]$
(b)	$H(\underline{\mathbf{p}}) \approx 0.9710$ when $\underline{\mathbf{p}} = [0.4, 0.6]$	<	$H(\underline{\mathbf{p}}) \approx 1.2955$ when $\underline{\mathbf{p}} = [0.1, 0.3, 0.6]$
(c)	$H(\underline{\mathbf{p}}) \approx 1.8464$ when $\underline{\mathbf{p}} = [0.1, 0.2, 0.3, 0.4]$	=	$H(\underline{\mathbf{p}}) \approx 1.8464$ when $\underline{\mathbf{p}} = [0.1, 0.3, 0.2, 0.4]$

First, note that the arguments of the entropy function are vectors. The vectors themselves represent probability mass functions. (Note that the sum of the elements in each vector is one.) Given a probability vector $\underline{\mathbf{p}}$,

$$H(\underline{\mathbf{p}}) \equiv - \sum_i p_i \log_2 p_i.$$

We can try to calculate the entropy values directly to compare them. This is done in the table above.

Alternatively, we can avoid directly computing the values by the following observation:

- a) When $\underline{\mathbf{p}} = [p_1, p_2]$, we have

$$H(\underline{\mathbf{p}}) \equiv -p_1 \log_2 p_1 - p_2 \log_2 p_2 = -p_1 \log_2 p_1 - (1 - p_1) \log_2 (1 - p_1).$$

This is the same as the binary entropy function $H(p)$ evaluated at $p = p_1$. In class, we have seen the plot of $H(p)$ as a function of p .

When the value of p is closer to 0.5, the entropy value is larger. Here, 0.4 is closer to 0.5 than 0.3 is.

- b) We are comparing $-0.4 \log_2 0.4 - 0.6 \log_2 0.6$ and $-0.1 \log_2 0.1 - 0.3 \log_2 0.3 - 0.6 \log_2 0.6$.

Both quantities have the term “ $-0.6 \log_2 0.6$ ”; so, we can ignore this term.

Now, note that $0.4 = 0.1 + 0.3$. So,

$$-0.4 \log_2 0.4 = -(0.1 + 0.3) \log_2 0.4 = -0.1 \log_2 0.4 - 0.3 \log_2 0.4.$$

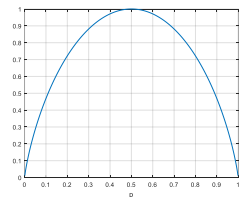
Because $\log_2(\cdot)$ is an increasing function, we know that $\log_2 0.4$ is greater than both $\log_2 0.1$ and $\log_2 0.3$.

With the negative signs, $-\log_2 0.4$ is less than both $-\log_2 0.1$ and $-\log_2 0.3$.

Therefore, continued from **●**, we have

$$-0.4 \log_2 0.4 = -0.1 \log_2 0.4 - 0.3 \log_2 0.4 < -0.1 \log_2 0.1 - 0.3 \log_2 0.3.$$

- c) The elements in vectors $\underline{\mathbf{p}}$ are the same on both sides except that the ordering is different. By the commutative property of addition, the resulting entropy values should be the same.



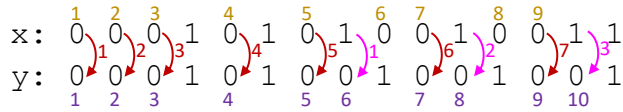
ECS 452: In-Class Exercise # 6 Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. [ENRE] Explanation is not required for this exercise.
3. Do not panic.

Date: 4 / 2 / 2020			
Name	ID <small>(last 3 digits)</small>		
Prapun	5	5	5

1. Consider a DMC whose samples of input and output are provided below



There are 15 samples (pairs of (x,y) values) here.

Estimate the following quantities:

a. \mathcal{X}

Channel input alphabet
= support of $X = \{0,1\}$

b. \mathcal{Y}

Channel output alphabet
= support of $Y = \{0,1\}$

c. $P[X=0] = \frac{9}{15} = \frac{3}{5} = 0.6$

Among the 15 samples, there are 9 samples with $x = 0$.

d. $p_Y(0) \equiv P[Y=0] = \frac{10}{15} = \frac{2}{3} \approx 0.6667$

Among the 15 samples, there are 10 samples with $y = 0$.

e. $\underline{p} = [p(0) \ p(1)] = \left[\frac{3}{5} \ \frac{2}{5} \right] = [0.6 \ 0.4]$

$$p(1) = 1 - p(0) = 1 - \frac{3}{5} = \frac{2}{5}$$

f. $\underline{q} = [q(0) \ q(1)] = \left[\frac{2}{3} \ \frac{1}{3} \right] \approx$

$[0.6667 \ 0.3333]$

$$q(1) = 1 - q(0) = 1 - \frac{2}{3} = \frac{1}{3}$$

g. $P[Y=0|X=0] = \frac{7}{9} \approx 0.7778$

Among the 9 samples with $x = 0$, there are 7 samples with $y = 0$.

h. $p_{Y|X}(1|0)$

$$= P[Y=1|X=0] = 1 - P[Y=0|X=0]$$

$$= 1 - \frac{7}{9} = \frac{2}{9} \approx 0.2222$$

i. $Q(0|1) = P[Y=0|X=1] = \frac{3}{6} = \frac{1}{2} = 0.5$

Among the 15 samples, there are 15 - 9 = 6 samples with $x = 1$.

Among the 6 samples with $x = 1$, there are 3 samples with $y = 0$.

j. Matrix \mathbf{Q}

Note that the Q matrix gives the conditional probabilities $P[Y=y|X=x]$. From earlier parts, we have three of such probabilities. The remaining one is

$$P[Y=1|X=1] = 1 - P[Y=0|X=1] = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$$

Therefore,

$$\mathbf{Q} = \begin{matrix} x \backslash y & 0 & 1 & x \backslash y & 0 & 1 \\ 0 & \left[\frac{7}{9} \ \frac{2}{9} \right] & \approx 0 & \left[\begin{matrix} 0.7778 & 0.2222 \end{matrix} \right] \\ 1 & \left[\frac{1}{2} \ \frac{1}{2} \right] & \approx 1 & \left[\begin{matrix} 0.5 & 0.5 \end{matrix} \right] \end{matrix}$$

k. $P[X=0, Y=0] = \frac{7}{15} \approx 0.4667$

Among the 15 samples, there are 7 samples with $x = 0$ and $y = 0$.

ECS 452: In-Class Exercise # 7 Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. [ENRE] Explanation is not required for this exercise.
3. [WASP] Write your answer(s) in the corresponding space(s) provided.
4. Do not panic.

Date: 7 / 2 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider a binary channel whose $Q(0|0)=0.7$ and $Q(0|1)=0.3$. Suppose $P[X=1]=0.4$. Find the channel matrix \mathbf{Q} , the output probability vector \mathbf{q} , and the joint pmf matrix \mathbf{P} .

$$P[X=0] = 1 - 0.4 = 0.6$$

The question explicitly specifies two elements in the \mathbf{Q} matrix.

The other two elements can be calculated by using the fact that, in the \mathbf{Q} matrix, any row sum should be 1.

Q matrix	P matrix																																				
<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="padding: 5px;">$x \backslash y$</td> <td style="padding: 5px; text-align: center;">0</td> <td style="padding: 5px; text-align: center;">1</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">0</td> <td style="padding: 5px; text-align: center;">0.7</td> <td style="padding: 5px; text-align: center;">0.3</td> <td style="padding: 5px; text-align: center;">$\times 0.6$</td> </tr> <tr> <td style="padding: 5px;">1</td> <td style="padding: 5px; text-align: center;">0.3</td> <td style="padding: 5px; text-align: center;">0.7</td> <td style="padding: 5px; text-align: center;">$\times 0.4$</td> </tr> </table>	$x \backslash y$	0	1		0	0.7	0.3	$\times 0.6$	1	0.3	0.7	$\times 0.4$	<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="padding: 5px;">$x \backslash y$</td> <td style="padding: 5px; text-align: center;">0</td> <td style="padding: 5px; text-align: center;">1</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">0</td> <td style="padding: 5px; text-align: center;">0.42</td> <td style="padding: 5px; text-align: center;">0.18</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">1</td> <td style="padding: 5px; text-align: center;">0.12</td> <td style="padding: 5px; text-align: center;">0.28</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;"></td> <td style="padding: 5px; text-align: center;">Σ</td> <td style="padding: 5px; text-align: center;">Σ</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;"></td> <td colspan="2" style="padding: 5px; text-align: center;">\mathbf{q}</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;"></td> <td colspan="2" style="padding: 5px; text-align: center;">[0.54 0.46]</td> <td style="padding: 5px;"></td> </tr> </table>	$x \backslash y$	0	1		0	0.42	0.18		1	0.12	0.28			Σ	Σ			\mathbf{q}				[0.54 0.46]		
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	[0.54 0.46]																																				

2. Consider a DMC whose channel matrix \mathbf{Q} is given below.

Suppose the input probability vector is $\mathbf{p} = [0.4, 0.3, 0.2, 0.1]$.

Calculate the missing values in the \mathbf{Q} matrix.

Then, find the output probability vector \mathbf{q} and the joint pmf matrix \mathbf{P} .

The missing values can be calculated by using the fact that, in the \mathbf{Q} matrix, any row sum should be 1.

Q matrix	P matrix																																																																														
<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="padding: 5px;">$x \backslash y$</td> <td style="padding: 5px; text-align: center;">1</td> <td style="padding: 5px; text-align: center;">2</td> <td style="padding: 5px; text-align: center;">3</td> <td style="padding: 5px; text-align: center;">4</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">1</td> <td style="padding: 5px; text-align: center;">0.2</td> <td style="padding: 5px; text-align: center;">0.2</td> <td style="padding: 5px; text-align: center;">0.3</td> <td style="padding: 5px; text-align: center;">0.3</td> <td style="padding: 5px; text-align: center;">$\times 0.4$</td> </tr> <tr> <td style="padding: 5px;">2</td> <td style="padding: 5px; text-align: center;">0.1</td> <td style="padding: 5px; text-align: center;">0.4</td> <td style="padding: 5px; text-align: center;">0.2</td> <td style="padding: 5px; text-align: center;">0.3</td> <td style="padding: 5px; text-align: center;">$\times 0.3$</td> </tr> <tr> <td style="padding: 5px;">3</td> <td style="padding: 5px; text-align: center;">0.3</td> <td style="padding: 5px; text-align: center;">0.1</td> <td style="padding: 5px; text-align: center;">0.1</td> <td style="padding: 5px; text-align: center;">0.5</td> <td style="padding: 5px; text-align: center;">$\times 0.2$</td> </tr> <tr> <td style="padding: 5px;">4</td> <td style="padding: 5px; text-align: center;">0.1</td> <td style="padding: 5px; text-align: center;">0.1</td> <td style="padding: 5px; text-align: center;">0.7</td> <td style="padding: 5px; text-align: center;">0.1</td> <td style="padding: 5px; text-align: center;">$\times 0.1$</td> </tr> </table>	$x \backslash y$	1	2	3	4		1	0.2	0.2	0.3	0.3	$\times 0.4$	2	0.1	0.4	0.2	0.3	$\times 0.3$	3	0.3	0.1	0.1	0.5	$\times 0.2$	4	0.1	0.1	0.7	0.1	$\times 0.1$	<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="padding: 5px;">$x \backslash y$</td> <td style="padding: 5px; text-align: center;">1</td> <td style="padding: 5px; text-align: center;">2</td> <td style="padding: 5px; text-align: center;">3</td> <td style="padding: 5px; text-align: center;">4</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">1</td> <td style="padding: 5px; text-align: center;">0.08</td> <td style="padding: 5px; text-align: center;">0.08</td> <td style="padding: 5px; text-align: center;">0.12</td> <td style="padding: 5px; text-align: center;">0.12</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">2</td> <td style="padding: 5px; text-align: center;">0.03</td> <td style="padding: 5px; text-align: center;">0.12</td> <td style="padding: 5px; text-align: center;">0.06</td> <td style="padding: 5px; text-align: center;">0.09</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">3</td> <td style="padding: 5px; text-align: center;">0.06</td> <td style="padding: 5px; text-align: center;">0.02</td> <td style="padding: 5px; text-align: center;">0.02</td> <td style="padding: 5px; text-align: center;">0.10</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;">4</td> <td style="padding: 5px; text-align: center;">0.01</td> <td style="padding: 5px; text-align: center;">0.01</td> <td style="padding: 5px; text-align: center;">0.07</td> <td style="padding: 5px; text-align: center;">0.01</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;"></td> <td style="padding: 5px; text-align: center;">Σ</td> <td style="padding: 5px; text-align: center;">Σ</td> <td style="padding: 5px; text-align: center;">Σ</td> <td style="padding: 5px; text-align: center;">Σ</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;"></td> <td colspan="4" style="padding: 5px; text-align: center;">\mathbf{q}</td> <td style="padding: 5px;"></td> </tr> <tr> <td style="padding: 5px;"></td> <td colspan="4" style="padding: 5px; text-align: center;">[0.18 0.23 0.27 0.32]</td> <td style="padding: 5px;"></td> </tr> </table>	$x \backslash y$	1	2	3	4		1	0.08	0.08	0.12	0.12		2	0.03	0.12	0.06	0.09		3	0.06	0.02	0.02	0.10		4	0.01	0.01	0.07	0.01			Σ	Σ	Σ	Σ			\mathbf{q}						[0.18 0.23 0.27 0.32]				
$x \backslash y$	1	2	3	4																																																																											
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ECS 452: In-Class Exercise # 8

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. [ENRE] Explanation is not required for this exercise.
3. [WASP] Write your answer(s) in the corresponding space(s) provided.
4. Do not panic.

Date: 14 / 2 / 2020			
Name			ID <small>(last 3 digits)</small>

1. Consider a DMC whose transition matrix \mathbf{Q} and joint pmf matrix \mathbf{P} are given below.

$$\mathbf{Q} = \begin{matrix} x \backslash y & 1 & 2 & 3 & 4 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.2 & 0.3 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.1 & 0.1 & 0.7 & 0.1 \end{bmatrix} \end{matrix}$$

$$\mathbf{P} = \begin{matrix} x \backslash y & 1 & 2 & 3 & 4 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.08 & 0.08 & 0.12 & 0.12 \\ 0.03 & 0.12 & 0.06 & 0.09 \\ 0.06 & 0.02 & 0.02 & 0.10 \\ 0.01 & 0.01 & 0.07 & 0.01 \end{bmatrix} \end{matrix}$$

We will consider four decoders:

- (a) **The naïve decoder.** For naïve decoder, $\hat{x}_{\text{naïve}}(y) = y$. Therefore, we can simply copy the values in the y -column into its decoding table.
- (b) **A DIY decoder defined by $\hat{x}_{\text{DIY}}(y) = 5 - y$.** The equation for the decoder is given; so we can simply plug-in each of the possible y values to get $\hat{x}_{\text{DIY}}(y)$.
- (c) **The MAP decoder.**
- (d) **The ML decoder.**

Deriving the MAP and ML decoder follows almost the same recipe: select the max value in each column and read the corresponding x -value. The difference is that the **MAP decoder uses the \mathbf{P} matrix** but the **ML decoder uses the \mathbf{Q} matrix**.

$$\mathbf{Q} = \begin{matrix} x \backslash y & 1 & 2 & 3 & 4 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.2 & 0.3 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.1 & 0.1 & 0.7 & 0.1 \end{bmatrix} \end{matrix}$$

$$\mathbf{P} = \begin{matrix} x \backslash y & 1 & 2 & 3 & 4 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.08 & 0.08 & 0.12 & 0.12 \\ 0.03 & 0.12 & 0.06 & 0.09 \\ 0.06 & 0.02 & 0.02 & 0.10 \\ 0.01 & 0.01 & 0.07 & 0.01 \end{bmatrix} \end{matrix}$$

Find the decoding table for each of these decoders. Also, find their probabilities of decoding error. Put your answers in the table below.

y	$\hat{x}_{\text{naïve}}(y)$	$\hat{x}_{\text{DIY}}(y)$	$\hat{x}_{\text{MAP}}(y)$	$\hat{x}_{\text{ML}}(y)$
1	1	4	1	3
2	2	3	2	2
3	3	2	1	4
4	4	1	1	3
$P(\mathcal{E})$	0.77	0.79	0.56	0.65

Once a decoder $\hat{x}(y)$ is defined, we can find its corresponding $P(\mathcal{E})$ easily from the \mathbf{P} matrix:

For each column y in the \mathbf{P} matrix, circle the element whose corresponding x value is the same as $\hat{x}(y)$; then, $P(\mathcal{C}) =$ the sum of the circled probabilities; and $P(\mathcal{E}) = 1 - P(\mathcal{C})$.

$$\mathbf{P} = \begin{matrix} \hat{x}_{\text{naïve}}(y) & 1 & 2 & 3 & 4 & \hat{x}_{\text{DIY}}(y) & 4 & 3 & 2 & 1 & \hat{x}_{\text{MAP}}(y) & 1 & 2 & 1 & 1 & \hat{x}_{\text{ML}}(y) & 3 & 2 & 4 & 3 \\ x \backslash y & 1 & 2 & 3 & 4 & x \backslash y & 1 & 2 & 3 & 4 & x \backslash y & 1 & 2 & 3 & 4 & x \backslash y & 1 & 2 & 3 & 4 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.08 & 0.08 & 0.12 & 0.12 \\ 0.03 & 0.12 & 0.06 & 0.09 \\ 0.06 & 0.02 & 0.02 & 0.10 \\ 0.01 & 0.01 & 0.07 & 0.01 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.08 & 0.08 & 0.12 & 0.12 \\ 0.03 & 0.12 & 0.06 & 0.09 \\ 0.06 & 0.02 & 0.02 & 0.10 \\ 0.01 & 0.01 & 0.07 & 0.01 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.08 & 0.08 & 0.12 & 0.12 \\ 0.03 & 0.12 & 0.06 & 0.09 \\ 0.06 & 0.02 & 0.02 & 0.10 \\ 0.01 & 0.01 & 0.07 & 0.01 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.08 & 0.08 & 0.12 & 0.12 \\ 0.03 & 0.12 & 0.06 & 0.09 \\ 0.06 & 0.02 & 0.02 & 0.10 \\ 0.01 & 0.01 & 0.07 & 0.01 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.08 & 0.08 & 0.12 & 0.12 \\ 0.03 & 0.12 & 0.06 & 0.09 \\ 0.06 & 0.02 & 0.02 & 0.10 \\ 0.01 & 0.01 & 0.07 & 0.01 \end{bmatrix} \end{matrix}$$

ECS 452: Exercise # 9

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. [ENRE] Explanation is not required for this exercise.
3. [WASP] Write your answer(s) in the corresponding space(s) provided.
4. Do not panic.

Date: 18 / 2 / 2019			
Name			ID (last 3 digits)

Consider a transmission of equally-likely codewords over a BSC using block encoding.

Suppose the crossover probability of the BSC is $p = 0.2$.

Use the provided MATLAB function `PE_minDist` to answer these problems.

```
function PE = PE_minDist(C,p)
% Function PE_minDist computes the error probability P(E) when code C
% is used for transmission over BSC with crossover probability p.
% Code C is defined by putting all its (valid) codewords as its rows.
M = size(C,1); % the number of (valid) codewords
k = log2(M);
n = size(C,2);

% Generate all possible n-bit received vectors
Y = dec2bin(0:2^n-1)-'0';

% Normally, we need to construct an extended Q matrix. However, because
% each conditional probability in there is a decreasing function of the
% Hamming distance, we can work with the distances instead of the
% conditional probability. In particular, instead of selecting the max in
% each column of the Q matrix, we consider min distance in each column.
dmin = zeros(1,2^n); % preallocation
for j = 1:(2^n)
    % for each received vector y,
    y = Y(j,:);
    % find the minimum distance
    % (the distance from y to the closest codeword)
    d = sum(mod(bsxfun(@plus,y,C),2),2);
    dmin(j) = min(d);
end

% From the distances, calculate the conditional probabilities.
% Note that we compute only the values that are to be selected (instead of
% calculating the whole Q first).
n1 = dmin; n0 = n-dmin;
Qmax = (p.^n1).*((1-p).^n0);
% Scale the conditional probabilities by the input probabilities and add
% the values. Note that we assume equally likely input.
PC = sum((1/M)*Qmax);
PE = 1-PC;
end
```

```
close all; clear all;

% ECS315 2019 Example 6.58
% ECS452 2019 Example 3.65
C = [0 0 0 0 0; 1 1 1 1 1]; % repetition code

p = 0.2;
PE_minDist(C,p)

>> PE_minDist_demo1
ans =
9.8506e-06
```

1. Suppose repetition code with $n = 5$ is used. Find the corresponding error probability $P(\mathcal{E})$.

In class, we have shown examples of how to use the function `PE_minDist` to find $P(\mathcal{E})$ for repetition code with $n = 5$. See, for example, the script `PE_minDist_demo1`. Here, the difference is simply that the crossover probability is changed from $1/100$ to 0.2 . So, by changing the value of p in the provided example to 0.2 , we can find the new $P(\mathcal{E})$.

$$P(\mathcal{E}) \approx 0.0579$$

2. Here, as in Problem 1, we will use $k = 1$ and $n = 5$. This implies that any code must contain two 5-bit codewords.

Assume that these two codewords are distinct.

- a. Give an example of a code that performs worse (gives larger $P(\mathcal{E})$) than repetition code.

Method 1: We can use trial and error.

Method 2: Alternatively, for BSC with $p < 0.5$, the transmitted bits are more likely to stay the same than to switch into their complements. This implies that the received vectors that have less bit errors are more likely to occur than ones with many bit errors. Therefore, code whose codewords are far away from each other seems to give small $P(\mathcal{E})$. When $n = 5$, the maximum "distance" between the codewords is 5. So, the repetition code in problem 1 already achieves this maximum. To find a code that performs worse, we try to make the distance between the two codewords smaller. This implies making some bits (in the corresponding positions) of the two codewords the same.

Some examples are provided below:

\mathcal{C}	Distance between the two codewords	$P(\mathcal{E})$
$\{0\ 0\ 0\ 0\ 0; 1\ 1\ 1\ 1\ 1\}$	5	0.0579
$\{0\ 0\ 0\ 0\ 0; 0\ 1\ 1\ 1\ 1\}$	4	0.1040
$\{0\ 0\ 0\ 0\ 0; 0\ 0\ 1\ 1\ 1\}$	3	0.1040
$\{0\ 0\ 0\ 0\ 0; 0\ 0\ 0\ 1\ 1\}$	2	0.2000
$\{0\ 0\ 0\ 0\ 0; 0\ 0\ 0\ 0\ 1\}$	1	0.2000

Note that smaller distance between the two codewords does not automatically correspond to smaller $P(\mathcal{E})$.

- b. Give an example of a code that is not a repetition code but performs as good as the repetition code. From the discussion in the previous part, it seems that, to be as good as the repetition code, we should try the code whose distance between the two codewords is 5. Such code can be found easily: set the first codeword to be any 5-bit binary vector; invert all the bits to get the second codeword. An example is given below:

$$\mathcal{C} = \{00111, 11000\}$$

3. In this problem, we will use $k = 2$ and $n = 5$.

This implies that any code must contain four 5-bit codewords.

Assume that these four codewords are distinct.

- a. Find the error probability $P(\mathcal{E})$ of the following code: $\mathcal{C} = \{00000, 10001, 01110, 11111\}$

$$P(\mathcal{E}) \approx 0.2832$$

- b. The best value of the error probability $P(\mathcal{E})$ for $k = 2$ and $n = 5$ is 0.2218.

Find at least one code that can achieve this $P(\mathcal{E})$.

Submit your answer of this part at <https://forms.gle/THbSRSqDWRpvg1329>.

You will get a penalty of “-1” if your answer is the same as the answer from another group who submitted before you. (Codes whose codewords are just “reordering” of another code will be

considered the same. For example, $\mathcal{C} = \{10001, 00000, 01110, 11111\}$ is considered the same as the code in part 3a.)

Submitted codes are published at <https://bit.ly/38tLsRe>.

Here are the solutions found in the submitted exercises:

- $[00110; 01011; 10000; 11101]$
- $[01100; 10101; 00010; 11011]$
- $[00000; 11001; 01110; 10111]$
- $[10100; 11111; 00001; 01010]$
- $[10010; 11001; 01100; 00111]$
- $[00011; 10000; 01101; 11110]$
- $[11010; 10001; 01101; 00110]$
- $[10101; 10010; 01100; 01011]$
- $[00111; 10001; 11010; 01100]$
- $[00101; 01000; 11110; 10011]$

Remark: It may be quite difficult to find one solution. However, with many groups working on the search, we expect some groups to be able to find solutions within reasonable time. Now, because the solutions must be posted on the web, other groups may look at the posted solutions and try to apply some “modification” to get their own solutions. The following operations do not change the distances between codewords and hence do not change the corresponding $P(\mathcal{E})$:

- Permute the columns
- Invert all the bits in some columns

For example, solution #2 can be obtained simply by permuting columns of solution #1.

Similarly, solutions #3 to #10 can be obtained from solution #1.

ECS 452: In-Class Exercise # 10

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 21 / 2 / 2019			
Name			ID <small>(last 3 digits)</small>

1. Consider two random variables X and Y whose joint pmf matrix is given by

$$\mathbf{P} = \begin{array}{c} x \setminus y \\ \begin{array}{cc} 1 & 2 \\ \hline 1 & \begin{bmatrix} 1/4 & 0 \end{bmatrix} \xrightarrow{\Sigma} 1/4 \\ 2 & \begin{bmatrix} 1/4 & 1/4 \end{bmatrix} \xrightarrow{\Sigma} 1/2 \\ 3 & \begin{bmatrix} 0 & 1/4 \end{bmatrix} \xrightarrow{\Sigma} 1/4 \\ \hline & \downarrow \Sigma \quad \downarrow \Sigma \\ & 1/2 \quad 1/2 \end{array} \end{array}$$

Calculate the following quantities.

a. $H(X, Y) = -\sum_{(x,y)} p(x, y) \log_2 p(x, y) = -4 \times \frac{1}{4} \times \log_2 \frac{1}{4} = 2$ [bits]

b. $H(X) = -\sum_x p(x) \log_2 p(x) = -2 \times \frac{1}{4} \times \log_2 \frac{1}{4} - \frac{1}{2} \times \log_2 \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$ [bits]

$p(x)$ is found by summing along each row of the \mathbf{P} matrix.

c. $H(Y) = -\sum_y q(y) \log_2 q(y) = -2 \times \frac{1}{2} \times \log_2 \frac{1}{2} = 1$ [bit]

$q(y)$ is found by summing along each column of the \mathbf{P} matrix.

Here, observe that Y is a uniform RV. Hence, alternatively, $H(Y) = \log_2 |\mathcal{Y}| = \log_2 2 = 1$ [bit].

d. $H(Y|X) = H(X, Y) - H(X) = 2 - 1.5 = 0.5$ [bits]

e. \mathbf{Q} matrix can be found by scaling each row of the \mathbf{P} matrix by $\frac{1}{p(x)}$.

$$\mathbf{Q} = \begin{array}{c} x \setminus y \\ \begin{array}{cc} 1 & 2 \\ \hline 1 & \begin{bmatrix} 1 & 0 \end{bmatrix} \xleftarrow{\times 4} 1 \\ 2 & \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \xleftarrow{\times 2} 2 \\ 3 & \begin{bmatrix} 0 & 1 \end{bmatrix} \xleftarrow{\times 4} 3 \\ \hline & \downarrow 1 \end{array} \end{array} = \begin{array}{c} x \setminus y \\ \begin{array}{cc} 1 & 2 \\ \hline 1 & \begin{bmatrix} 1/4 & 0 \end{bmatrix} \\ 2 & \begin{bmatrix} 1/4 & 1/4 \end{bmatrix} \\ 3 & \begin{bmatrix} 0 & 1/4 \end{bmatrix} \end{array} = \mathbf{P}$$

f. $H(Y|X = 2) = -\sum_y Q(y|2) \log_2 Q(y|2) = -2 \times \frac{1}{2} \times \log_2 \frac{1}{2} = 1$ [bit]

Note that we use the “ $x = 2$ ” row of the \mathbf{Q} matrix to calculate this conditional entropy.

Remark: We can also find $H(Y|X = 1) = H(Y|X = 3) = 0$. Therefore, we can double-check our answer in part d: $H(Y|X) = \sum_x H(Y|x)p(x) = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{2}\right) + \left(0 \times \frac{1}{4}\right) = \frac{1}{2}$.

ECS 452: In-Class Exercise # 11

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. [ENRE] Explanation is not required for this exercise
3. **Do not panic.**

Date: 28 / 2 / 2019			
Name			ID <small>(last 3 digits)</small>

1. For each of the following DMC's probability transition matrices \mathbf{Q} , (i) indicate whether the corresponding DMC is symmetric (Yes or No), (ii) indicate whether the corresponding DMC is weakly symmetric (Yes or No), (iii) evaluate the corresponding capacity value (your answer should be of the form X.XXXX).

To be symmetric DMC, the Q matrix must satisfy two conditions:

- 1 rows are permutations of each other
- 2 columns are permutations of each other

To be weakly symmetric DMC, the Q matrix must satisfy two conditions:

- 1 rows are permutations of each other
- 2 same column sum

The (informational) channel capacity is

$$C = \max_{\mathbf{p}} I(\mathbf{p}, \mathbf{Q})$$

However, in class, we have seen that there are multiple special cases which have easier formulas.

Q	Symmetric Channel?	Weakly Symmetric Channel?	C
$\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$	1 ✓ 2 ✓ Yes	1 ✓ 2 ✓ Yes	The DMC is a BSC with $p = 0.25$. $C = 1 - H(p) = 1 - H(0.25)$ $\approx 1 - 0.8113 = 0.1887$.
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$	1 ✗ 2 ✗ No	1 ✗ 2 ✗ No	The DMC satisfies the condition for NO ² . (Each column of Q has only one non-zero element.) $C = \log_2 \mathcal{X} = \log_2 2 = 1.000$.
$\begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$	1 ✓ 2 ✓ Yes	1 ✓ 2 ✓ Yes	The DMC is weakly symmetric. $C = \log_2 \mathcal{Y} - H(\mathbf{r})$ $= \log_2 3 - H\left(\left[\begin{smallmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \end{smallmatrix}\right]\right)$ $\approx 1.5850 - 1.5 = 0.0850$.
$\begin{bmatrix} 0.75 & 0.25 & 0 & 0 \\ 0 & 0.25 & 0 & 0.75 \\ 0 & 0.25 & 0.75 & 0 \end{bmatrix}$	1 ✓ 2 ✗ No	1 ✓ 2 ✓ Yes	The DMC is weakly symmetric. $C = \log_2 \mathcal{Y} - H(\mathbf{r})$ $= \log_2 4 - H\left(\left[\begin{smallmatrix} 3 & 1 \\ 4 & 4 \end{smallmatrix}\right]\right)$ $\approx 2 - 0.8113 = 1.1887$.

$$H\left(\left[\begin{smallmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \end{smallmatrix}\right]\right) = -2 \times \frac{1}{4} \log_2 \frac{1}{4} - \frac{1}{2} \log_2 \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$$