ECS 452: In-Class Exercise # 12

Instructions

1. Separate into groups of no more than three students each. Only one submission is needed for each group.
2. **[ENRE] Explanation is not required for this exercise**
3. **Do not panic.**

Date: 10 / 3 / 2020			
Name	ID (last 3 digits)		

Assume GF(2).

1. Calculate the following quantities:

a. $1 \oplus 1 = 0$

b. $0 \oplus 1 \oplus 1 = (0 \oplus 1) \oplus 1 = 1 \oplus 1 = 0$

c. $1 \cdot 0 = 0$

d. $1 \cdot 0 \cdot 1 = (1 \cdot 0) \cdot 1 = 0 \cdot 1 = 0$

e. $[0 \quad 1 \quad 0] \oplus [1 \quad 1 \quad 1] = [0 \oplus 1 \quad 1 \oplus 1 \quad 0 \oplus 1] = [1 \quad 0 \quad 1]$

f. $[0 \quad 1 \quad 1]\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = [(0 \cdot 1) \oplus (1 \cdot 1) \oplus (1 \cdot 0) \quad (0 \cdot 0) \oplus (1 \cdot 1) \oplus (1 \cdot 1)] = [0 \oplus 1 \oplus 0 \quad 0 \oplus 1 \oplus 1] = [1 \quad 0]$

Alternatively, multiplying by $[0 \quad 1 \quad 1]$ means we simply add the last two rows of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

2. Fill in the blanks:

$[\_ \quad \_ \quad \_] \oplus [1 \quad 0 \quad 1] = [0 \quad 0 \quad 0]$

$[\_ \quad \_ \quad \_] = [1 \quad 0 \quad 1]$

$\underline{\mathbf{a}} \oplus \underline{\mathbf{b}} = \underline{0}$

$\underline{\mathbf{a}} \oplus \underline{\mathbf{b}} \oplus \underline{\mathbf{b}} = \underline{0} \oplus \underline{\mathbf{b}}$

$\underline{0}$

$\underline{\mathbf{a}} = \underline{\mathbf{b}}$

If we define the "negative" of $\underline{\mathbf{b}}$ to be the vector that give $\underline{\mathbf{0}}$ when added to $\underline{\mathbf{b}}$. Here, in GF(2), we see that "negative" of $\underline{\mathbf{b}}$ is $\underline{\mathbf{b}}$ itself.

3. Consider a matrix $\mathbf{G}$. Suppose $[0 \quad 1]\mathbf{G} = [1 \quad 0 \quad 1]$ and $[1 \quad 0]\mathbf{G} = [0 \quad 0 \quad 1]$. Find $\mathbf{G}$.

First, recall that

$$[\quad]_{m \times t}[\quad]_{t \times n} = [\quad]_{m \times n}.$$

Here, $m = 1$, $t = 2$, and $n = 3$. So, $\mathbf{G}$ is a 2×3 matrix.

Let's write $\boldsymbol{G} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then,

$$[0 \quad 1]\mathbf{G} = [0 \quad 1]\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = [d \quad e \quad f],$$

and

$$[1 \quad 0]\mathbf{G} = [1 \quad 0]\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = [a \quad b \quad c].$$

From the provided information, we can conclude that

$$[d \quad e \quad f] = [1 \quad 0 \quad 1] \text{ and } [a \quad b \quad c] = [0 \quad 0 \quad 1].$$

Therefore, $\mathbf{G} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

ECS 452: In-Class Exercise # 13

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former group after the midterm.**
2. Only one submission is needed for each group.
3. **[ENRE] Explanation is not required for this exercise**
4. **Do not panic.**

Date: 13 / 3 / 2020			
Name	ID (last 3 digits)		

1. For each code given below, check whether the code is linear. Note that these codes are from Exercise 9. They are all optimal codes (minimum $P(\mathcal{E})$) for the case when $n = 5$ and $k = 2$.

$\mathcal{C}$	Linear?	Reason
{00110, 01011, 10000, 11101}	No	$\underline{\mathbf{0}} \notin \mathcal{C}$.
{01100, 10101, 00010, 11011}	No	$\underline{\mathbf{0}} \notin \mathcal{C}$.
{00000, 11001, 01110, 10111} $\underline{\mathbf{c}}^{(1)}, \underline{\mathbf{c}}^{(2)}, \underline{\mathbf{c}}^{(3)}, \underline{\mathbf{c}}^{(4)}$	Yes	$\underline{\mathbf{0}} \in \mathcal{C}$. So, we have to check that the sum of any non-zero codewords is still a codeword. Here, $\underline{\mathbf{c}}^{(2)} \oplus \underline{\mathbf{c}}^{(3)} = 11001 \oplus 01110 = 10111 = \underline{\mathbf{c}}^{(4)}$. Therefore, we also have $\underline{\mathbf{c}}^{(2)} \oplus \underline{\mathbf{c}}^{(4)} = \underline{\mathbf{c}}^{(3)}$ and $\underline{\mathbf{c}}^{(3)} \oplus \underline{\mathbf{c}}^{(4)} = \underline{\mathbf{c}}^{(2)}$.
{10100, 11111, 00001, 01010}	No	$\underline{\mathbf{0}} \notin \mathcal{C}$.
{10010, 11001, 01100, 00111}	No	$\underline{\mathbf{0}} \notin \mathcal{C}$.

2. A linear block code uses the following generator matrix $\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

   a. Find the codeword length n = number of columns of $\mathbf{G}$ = 4

   b. Find the codeword for the message $\underline{\mathbf{b}} = [1 \quad 0]$

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = [1 \quad 0]\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = [1 \quad 0 \quad 0 \quad 1]$$

   Alternatively,

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = [1 \quad 0]\begin{bmatrix} \underline{\mathbf{g}}^{(1)} \\ \underline{\mathbf{g}}^{(2)} \end{bmatrix} \overset{\text{block matrix multiplication}}{=} \underline{\mathbf{g}}^{(1)} = [1 \quad 0 \quad 0 \quad 1]$$

   c. Find the codebook for this code.

Codebook can be generated by working **row-wise**: generating each codeword one-by-one

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = [b_1 \quad b_2]\begin{bmatrix} \underline{\mathbf{g}}^{(1)} \\ \underline{\mathbf{g}}^{(2)} \end{bmatrix} \overset{\text{block matrix multiplication}}{=} b_1\underline{\mathbf{g}}^{(1)} \oplus b_2\underline{\mathbf{g}}^{(2)}$$

$\underline{\mathbf{b}}$	$\underline{\mathbf{x}}$
00	$0\underline{\mathbf{g}}^{(1)} \oplus 0\underline{\mathbf{g}}^{(2)} = \underline{\mathbf{0}} = 0000$
01	$0\underline{\mathbf{g}}^{(1)} \oplus 1\underline{\mathbf{g}}^{(2)} = \underline{\mathbf{g}}^{(2)} = 1110$
10	$1\underline{\mathbf{g}}^{(1)} \oplus 0\underline{\mathbf{g}}^{(2)} = \underline{\mathbf{g}}^{(1)} = 1001$
11	$1\underline{\mathbf{g}}^{(1)} \oplus 1\underline{\mathbf{g}}^{(2)} = \underline{\mathbf{g}}^{(1)} \oplus \underline{\mathbf{g}}^{(2)} = 0111$

$b_1 b_2$

Alternatively, we can work **column-wise**:

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = [b_1 \quad b_2]\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = [b_1 \oplus b_2 \quad b_2 \quad b_2 \quad b_1]$$

The first bit is the sum of b_1 and b_2.

The second bit and the third bit are simply b_2.

The fourth bit is simply b_1.

ECS 452: In-Class Exercise # 14

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Only one submission is needed for each group.
3. **Do not panic.**

Date: 24 / 3 / 2020			
Name	ID (last 3 digits)		

1. A codeword $[0 \quad 1 \quad 1 \quad 0]$ is sent over the BSC. Suppose the error pattern is $\underline{\mathbf{e}} = [1 \quad 1 \quad 0 \quad 1]$. Find the observed vector at the receiver.

$$\underline{\mathbf{y}} = \underline{\mathbf{x}} \oplus \underline{\mathbf{e}} = [0110] \oplus [1101] = [1011]$$

Alternatively, from the error pattern, we know that the all bits of $\underline{\mathbf{x}}$ will be flipped by the BSC except the third bit. So, $\underline{\mathbf{y}}$ is constructed accordingly.

2. A codeword $[0 \quad 1 \quad 1 \quad 0]$ is sent over the BSC. Suppose the observed vector at the receiver is $\underline{\mathbf{y}} = [1 \quad 0 \quad 0 \quad 1]$. Find the error pattern.

$$\underline{\mathbf{y}} = \underline{\mathbf{x}} \oplus \underline{\mathbf{e}}$$
$$\underline{\mathbf{x}} \oplus \underline{\mathbf{y}} = \underbrace{\underline{\mathbf{x}} \oplus \underline{\mathbf{x}}}_{0} \oplus \underline{\mathbf{e}}$$
$$\underline{\mathbf{x}} \oplus \underline{\mathbf{y}} = \underline{\mathbf{e}}$$

From $\underline{\mathbf{y}} = \underline{\mathbf{x}} \oplus \underline{\mathbf{e}}$, we have $\underline{\mathbf{e}} = \underline{\mathbf{x}} \oplus \underline{\mathbf{y}} = [0110] \oplus [1001] = [1111]$

Alternatively, recall that the error pattern indicates the locations of error. Here, all bits of $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are different. So, the error pattern should be [1111].

3. A codeword is sent over the BSC. Suppose the observed vector at the receiver is $\underline{\mathbf{y}} = [0 \quad 0 \quad 1 \quad 0]$ and the error pattern is $\underline{\mathbf{e}} = [1 \quad 1 \quad 1 \quad 0]$. Find the transmitted codeword.

$$\underline{\mathbf{y}} = \underline{\mathbf{x}} \oplus \underline{\mathbf{e}}$$
$$\underline{\mathbf{y}} \oplus \underline{\mathbf{e}} = \underline{\mathbf{x}} \oplus \underbrace{\underline{\mathbf{e}} \oplus \underline{\mathbf{e}}}_{0}$$
$$\underline{\mathbf{y}} \oplus \underline{\mathbf{e}} = \underline{\mathbf{x}}$$

From $\underline{\mathbf{y}} = \underline{\mathbf{x}} \oplus \underline{\mathbf{e}}$, we have $\underline{\mathbf{x}} = \underline{\mathbf{y}} \oplus \underline{\mathbf{e}} = [0010] \oplus [1110] = [1100]$

Alternatively, the error pattern says that all bits are received incorrectly except the 4^{th} bit. Therefore, to recover $\underline{\mathbf{x}}$, we need to flip all bits of $\underline{\mathbf{y}}$ except the 4^{th} bit.

ECS 452: In-Class Exercise # 15

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Only one submission is needed for each group.
3. **Do not panic.**

Date: 27 / 3 / 2020			
Name	ID (last 3 digits)		

1. Consider a linear block code whose generator matrix is

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

a. Find the length k of each message block

G has 3 rows. Therefore, $k = 3$.

b. Find the code length n

G has 5 columns. Therefore, $n = 5$.

c. In the table below, list all possible data (message) vectors $\underline{\mathbf{b}}$ in the leftmost column (one in each row). Then, find the corresponding codewords $\underline{\mathbf{x}}$ and their weights in the second and third columns, respectively.

$\underline{\mathbf{b}}$	$\underline{\mathbf{x}}$	$w(\underline{\mathbf{x}})$
b_1 b_2 b_3	x_1 x_2 x_3 x_4 x_5	
0 0 0	0 0 0 0 0	0
0 0 1	0 1 0 1 0	2
0 1 0	0 1 1 0 1	3
0 1 1	0 0 1 1 1	3
1 0 0	1 0 0 0 1	2
1 0 1	1 1 0 1 1	4
1 1 0	1 1 1 0 0	3
1 1 1	1 0 1 1 0	3

First, we list all possible $\underline{\mathbf{b}}$.

Next, from $\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G}$, we can calculate the codeword $\underline{\mathbf{x}}$ corresponding to each $\underline{\mathbf{b}}$ one by one. Alternatively, by considering $\underline{\mathbf{b}} = [b_1 b_2 b_3]$ and carrying out the multiplication $\underline{\mathbf{x}} = [b_1 b_2 b_3]\mathbf{G}$, we have

$$\underline{\mathbf{x}} = [b_1, b_2 \oplus b_3, b_2, b_3, b_1 \oplus b_2].$$

So, each "column" of the answer for $\underline{\mathbf{x}}$ can be calculated accordingly. In particular,

- the 1st, 3rd, and 4th columns are simply copied from the columns for b_1, b_2, and b_3 respectively,
- the 2nd column is simply the sum of the columns for b_2 and b_3
- the 5th column is simply the sum of the columns for b_1 and b_2.

d. Find the minimum distance $d_{\min}$ for this code.

Because the code is linear,

$$d_{\min} = \min_{\underline{\mathbf{x}} \neq \underline{\mathbf{0}}} w(\underline{\mathbf{x}}) = 2.$$

e. What is the maximum number of bit errors that this code can guarantee to **detect**?

$$d_{\min} - 1 = 1$$

f. What is the maximum number of bit errors that this code can guarantee to **correct**?

$$\left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0$$

2. Consider a linear block code whose generator matrix is

$$\mathbf{G} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Suppose the minimum distance $d_{\min}$ for this code is $d_{\min} = 8$.

a. What is the maximum number of bit errors that this code can guarantee to **detect**?

$$d_{\min} - 1 = 7$$

b. What is the maximum number of bit errors that this code can guarantee to **correct**?

$$\left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor = \left\lfloor \frac{7}{2} \right\rfloor = 3$$

ECS 452: In-Class Exercise # 16

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Only one submission is needed for each group.
3. [**ENRE**] Explanation is not required for this exercise.
4. **Do not panic.**

Date: 31 / 3 / 2020			
Name	ID (last 3 digits)		

1. Consider a linear block code that uses *parity checking on a square array*:

First, we use the provided definition to write down the equations that produce the parity bits. This definition is exactly the same as the one given in lecture when we defined parity checking on a square array

b_1	b_3	p_1
b_2	b_4	p_2
p_3	p_4	

$p_1 = b_1 \oplus b_3$

$p_2 = b_2 \oplus b_4$

$p_3 = b_1 \oplus b_2$

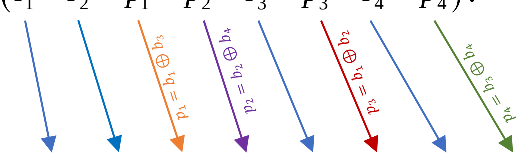
$p_4 = b_3 \oplus b_4$

Each parity bit p_i is calculated such that the corresponding row or column has even parity.
Suppose the following bits arrangement is used in the codeword:

$$\underline{\mathbf{x}} = \begin{pmatrix} b_1 & b_2 & p_1 & p_2 & b_3 & p_3 & b_4 & p_4 \end{pmatrix}.$$

a. Find the generator matrix $\mathbf{G}$.

Recall that the 1s and 0s in the j^{th} column of **G** tells which positions of the data bits are combined ($\oplus$) to produce the j^{th} bit in the codeword.

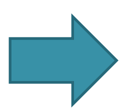


$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

b. Find the codeword for the message $\underline{\mathbf{b}} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$.

Method 1: First, we fill out the array above with the message. Then, we calculate the parity bits.

1	1	p_1
0	0	p_2
p_3	p_4	



1	1	0
0	0	0
1	1	

The codeword can be read directly from the array: $\underline{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$.

Method 2: It is still true that $\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G}$. Therefore, we can still use our old technique: to find $\underline{\mathbf{x}}$ when $\underline{\mathbf{b}} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$, we simply need to add the first and the third rows of $\mathbf{G}$. This also gives $\underline{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$.

c. Find the parity-check matrix **H**.

We look at two parts of **G**: the message part and the parity part.

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The parity part (columns) from **G** is transposed and put into the message positions (columns). The remaining columns are filled in by an identity matrix.

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

ECS 452: In-Class Exercise # 17

Instructions

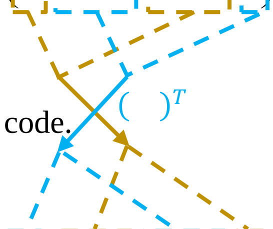
1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Only one submission is needed for each group.
3. **Do not panic.**

Date: 3 / 4 / 2020			
Name	ID (last 3 digits)		

Consider a block code whose generator matrix is

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

a. Find the parity check matrix **H** of this code.



The crossing here tries to capture the fact that there is a swapping of the positions.

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

b. Suppose we receive $\underline{\mathbf{y}}$ = 011111.

i. Find the syndrome vector $\underline{\mathbf{s}}$.

Because the 1s in $\underline{\mathbf{y}}$ are in the last five positions, to find the syndrome, we add the last five columns of **H**.

$$\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

ii. Find the decoded codeword $\hat{\underline{\mathbf{x}}}$.

The syndrome $\underline{\mathbf{s}}$ is the same as the ***last*** column of $\mathbf{H}$.

Therefore, $\hat{\underline{\mathbf{e}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and

$$\hat{\underline{\mathbf{x}}} = \underline{\mathbf{y}} - \hat{\underline{\mathbf{e}}} = \underline{\mathbf{y}} \oplus \hat{\underline{\mathbf{e}}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

iii. Find the decoded message $\hat{\underline{\mathbf{b}}}$.

From $\mathbf{G}$, we have columns of $\mathbf{I}_3$ in the 1st, 4th, and 5th columns; so, given a codeword $\underline{\mathbf{x}}$, the message $\underline{\mathbf{b}}$ corresponding to this codeword is given by the codeword's 1st, 4th, and 5th bits. Here, the decoded codeword is $\hat{\underline{\mathbf{x}}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$. Therefore, the corresponding decoded message is $\hat{\underline{\mathbf{b}}} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$.

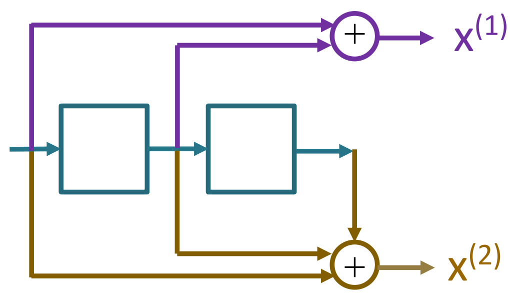
ECS 452: In-Class Exercise # 18

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Only one submission is needed for each group.
3. [**ENRE**] Explanation is not required for this exercise.
4. **Do not panic.**

Date: 7 / 4 / 2020			
Name	ID (last 3 digits)		

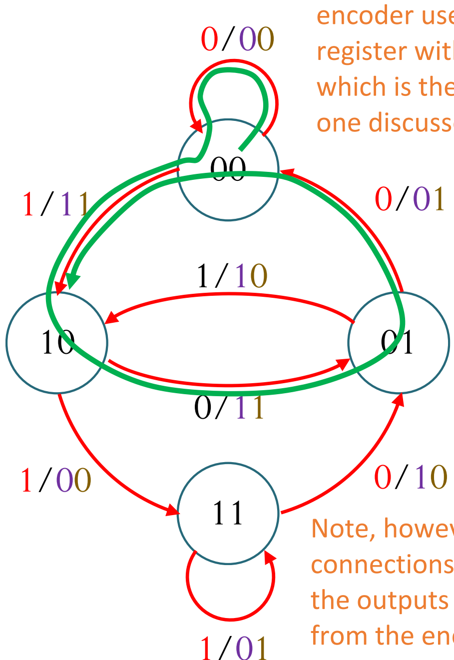
Consider a convolution encoder represented by the following diagram



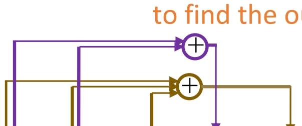
(a) Draw the corresponding state (transition) diagram

First, observe that this encoder uses a shift register with two FFs which is the same as the one discussed in lecture.

Therefore, the arrows will be the same as what we had in the lecture.



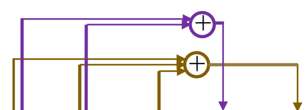
Note, however, that the connections that produce the outputs are different from the encoder in lecture. Therefore, we simply need to find the outputs.



b	s_1	s_2	$x^{(1)}$	$x^{(2)}$
0	0	0	0	0
1	0	0	1	1
0	0	1	0	1
1	0	1	1	0
0	1	0	1	1
1	1	0	0	0
0	1	1	1	0
1	1	1	0	1

(b) Suppose the information bits (the message bits) are $\underline{\mathbf{b}}$ = 01001.
Find the corresponding codeword $\underline{\mathbf{x}}$

i. by using the direct method (filling out the table below) without the help of the state diagram from part (a).



b	s_1	s_2	$x^{(1)}$	$x^{(2)}$
0	0	0	0	0
1	0	0	1	1
0	1	0	1	1
0	0	1	0	1
1	0	0	1	1

Note that the final output is one row vector resulting from interleaving the upper and lower outputs.

$\underline{\mathbf{x}}$ = 0011110111

and

ii. by "tracing" the corresponding path on the state diagram derived in part (a)
Draw/highlight your trace on the state diagram in part (a) using different pen color.

See the trace in the diagram on the left.

$\underline{\mathbf{x}}$ = 0011110111

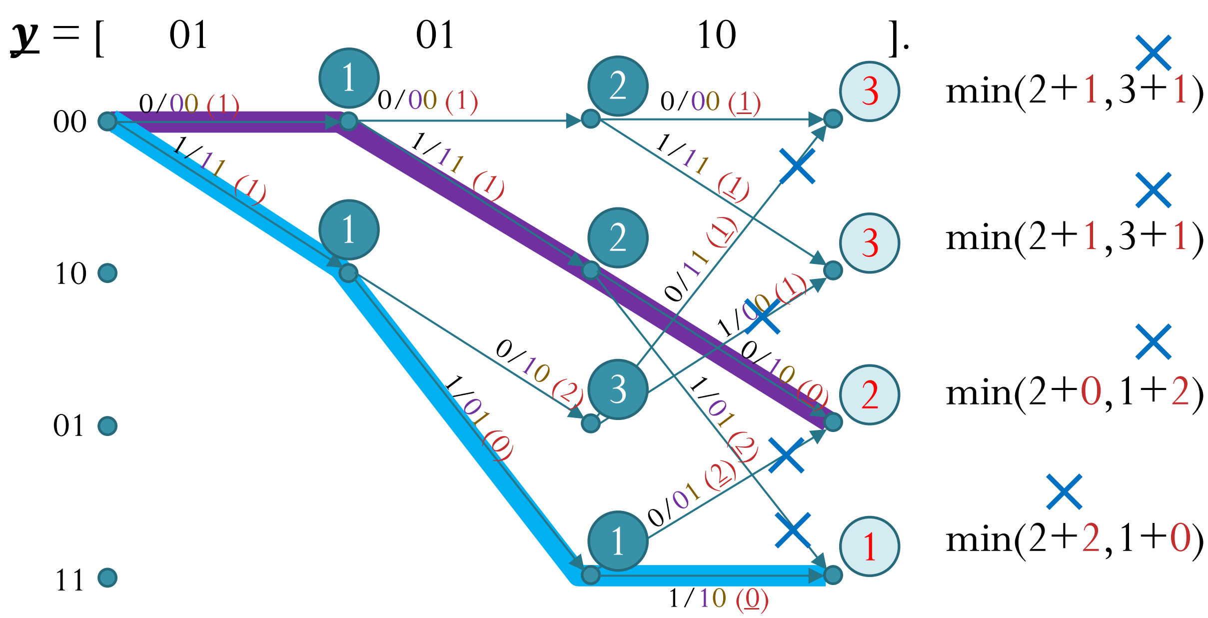
ECS 452: In-Class Exercise # 19

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Only one submission is needed for each group.
3. [**ENRE**] Explanation is not required for this exercise.
4. **Do not panic.**

Date: 10 / 4 / 2020			
Name	ID (last 3 digits)		

Consider a convolutional encoder whose trellis diagram is given below.
Vector $\underline{\mathbf{y}}$ and the numbers enclosed by the round brackets ( ) are used in the second problem.



1. Suppose the data vector is $\underline{\mathbf{b}}$ = [010]. Find the corresponding codeword $\underline{\mathbf{x}}$.

   Reading from the **purple path**, we get [001110]

2. Suppose that we observe $\underline{\mathbf{y}}$ = 010110… at the input of the minimum distance decoder.

   The decoder uses **Viterbi's algorithm**. (The first two stages were already calculated for you.) Your job is to work on the last stage.

   a. Write down
      (1) all the (distance) values on the branches and
      (2) the (chosen) cumulative distance values inside all the circles
      in the figure above.

   b. Put "x" on the branches that are removed by the Viterbi algorithm.

   c. Suppose the decoder works only on the first six bits of $\underline{\mathbf{y}}$.

      Find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$.

      Reading from the **blue path**, we get [110110].

      $\hat{\underline{\mathbf{x}}}$ = [110110] $\hat{\underline{\mathbf{b}}}$ = [111].

ECS 452: In-Class Exercise # 20

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 14 / 4 / 2020			
Name	ID (last 3 digits)		

1. Consider three vectors: $\vec{\mathbf{v}}^{(1)} = \begin{pmatrix} -5 \\ 5 \\ -1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{v}}^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, and $\vec{\mathbf{v}}^{(3)} = \begin{pmatrix} 3 \\ -3 \\ 1 \\ -1 \end{pmatrix}$. Let $\vec{\mathbf{e}}^{(1)} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{e}}^{(2)} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

   a. Show that $\vec{\mathbf{e}}^{(1)}$ and $\vec{\mathbf{e}}^{(2)}$ are orthonormal.

   To show they are orthonormal, we need to show that they are orthogonal and have unit norm.

   $\| \vec{\mathbf{x}} \|^2 = \sum_i x_i^2$

   $$\| \vec{\mathbf{e}}^{(1)} \|^2 = \left(\tfrac{1}{2}\right)^2 \left((-1)^2 + 1^2 + 1^2 + (-1)^2\right) = \tfrac{1}{4}(4) = 1 \Rightarrow \| \vec{\mathbf{e}}^{(1)} \| = 1$$

   $$\| \vec{\mathbf{e}}^{(2)} \|^2 = \left(\tfrac{1}{2}\right)^2 \left(1^2 + (-1)^2 + 1^2 + (-1)^2\right) = \tfrac{1}{4}(4) = 1 \Rightarrow \| \vec{\mathbf{e}}^{(2)} \| = 1$$

   Both have unit norm.

   $$\langle \vec{\mathbf{e}}^{(1)}, \vec{\mathbf{e}}^{(2)} \rangle = \tfrac{1}{2} \times \tfrac{1}{2} \times \left((-1)(1) + (1)(-1) + (1)(1) + (-1)(-1)\right) = 0 \Rightarrow$$ They are orthogonal.

   $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = \sum_i x_i y_i$

   b. Calculate the following inner products:

   $$\langle \vec{\mathbf{v}}^{(1)}, \vec{\mathbf{e}}^{(1)} \rangle = \begin{array}{rrrrrl} & -5 & 5 & -1 & 1 & \times \\ \frac{1}{2}(& -1 & 1 & 1 & -1 &) \\ \hline \frac{1}{2}(& 5 & 5 & -1 & -1 &) \xrightarrow{\Sigma} \frac{8}{2} = 4 \end{array}$$

   $$\langle \vec{\mathbf{v}}^{(1)}, \vec{\mathbf{e}}^{(2)} \rangle = \begin{array}{rrrrrl} & -5 & 5 & -1 & 1 & \times \\ \frac{1}{2}(& 1 & -1 & 1 & -1 &) \\ \frac{1}{2}(& -5 & -5 & -1 & -1 &) \xrightarrow{\Sigma} \frac{-12}{2} = -6 \end{array}$$

   $$\langle \vec{\mathbf{v}}^{(2)}, \vec{\mathbf{e}}^{(1)} \rangle = \begin{array}{rrrrrl} & -1 & 1 & 0 & 0 & \times \\ \frac{1}{2}(& -1 & 1 & 1 & -1 &) \\ \hline \frac{1}{2}(& 1 & 1 & 0 & 0 &) \xrightarrow{\Sigma} \frac{2}{2} = 1 \end{array}$$

   $$\langle \vec{\mathbf{v}}^{(2)}, \vec{\mathbf{e}}^{(2)} \rangle = \begin{array}{rrrrrl} & -1 & 1 & 0 & 0 & \times \\ \frac{1}{2}(& 1 & -1 & 1 & -1 &) \\ \frac{1}{2}(& -1 & -1 & 0 & 0 &) \xrightarrow{\Sigma} \frac{-2}{2} = -1 \end{array}$$

   $$\langle \vec{\mathbf{v}}^{(3)}, \vec{\mathbf{e}}^{(1)} \rangle = \begin{array}{rrrrrl} & 3 & -3 & 1 & -1 & \times \\ \frac{1}{2}(& -1 & 1 & 1 & -1 &) \\ \hline \frac{1}{2}(& -3 & -3 & 1 & 1 &) \xrightarrow{\Sigma} \frac{-4}{2} = -2 \end{array}$$

   $$\langle \vec{\mathbf{v}}^{(3)}, \vec{\mathbf{e}}^{(2)} \rangle = \begin{array}{rrrrrl} & 3 & -3 & 1 & -1 & \times \\ \frac{1}{2}(& 1 & -1 & 1 & -1 &) \\ \frac{1}{2}(& 3 & 3 & 1 & 1 &) \xrightarrow{\Sigma} \frac{8}{2} = 4 \end{array}$$

   c. Suppose we use $\vec{\mathbf{e}}^{(1)}$ and $\vec{\mathbf{e}}^{(2)}$ as the new axes. Find the corresponding vectors $\vec{\mathbf{c}}^{(1)}, \vec{\mathbf{c}}^{(2)}$, and $\vec{\mathbf{c}}^{(3)}$ that represent $\vec{\mathbf{v}}^{(1)}, \vec{\mathbf{v}}^{(2)}$, and $\vec{\mathbf{v}}^{(3)}$ in the new coordinate system defined by $\vec{\mathbf{e}}^{(1)}$ and $\vec{\mathbf{e}}^{(2)}$.

   We use the inner products that we calculated in the previous part.

   $$\vec{\mathbf{c}}^{(1)} = \begin{pmatrix} \langle \vec{\mathbf{v}}^{(1)}, \vec{\mathbf{e}}^{(1)} \rangle \\ \langle \vec{\mathbf{v}}^{(1)}, \vec{\mathbf{e}}^{(2)} \rangle \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \quad \vec{\mathbf{c}}^{(2)} = \begin{pmatrix} \langle \vec{\mathbf{v}}^{(2)}, \vec{\mathbf{e}}^{(1)} \rangle \\ \langle \vec{\mathbf{v}}^{(2)}, \vec{\mathbf{e}}^{(2)} \rangle \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{\mathbf{c}}^{(3)} = \begin{pmatrix} \langle \vec{\mathbf{v}}^{(3)}, \vec{\mathbf{e}}^{(1)} \rangle \\ \langle \vec{\mathbf{v}}^{(3)}, \vec{\mathbf{e}}^{(2)} \rangle \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

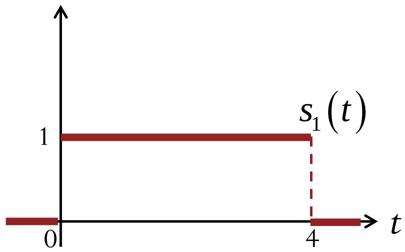
ECS 452: In-Class Exercise # 21

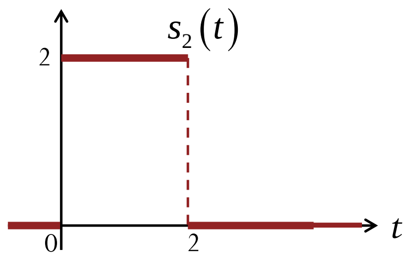
Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 17 / 4 / 2020			
Name	ID (last 3 digits)		

1. Consider two waveforms $s_1(t)$ and $s_2(t)$ shown below.





a. Find the energy of each waveform.

$E_1 = E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2\,dt = \int_0^4 |1|^2\,dt = 4.$

$E_2 = E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2\,dt = \int_0^2 |2|^2\,dt = 4 \times 2 = 8.$

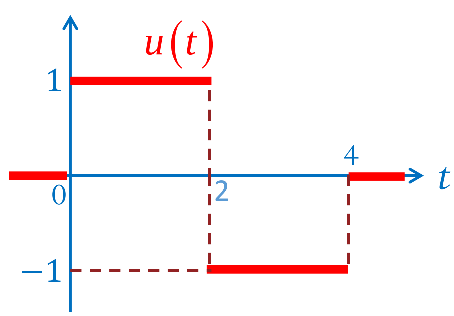
b. Calculate their inner product $\langle s_1(t), s_2(t)\rangle$.

$$\langle s_1(t), s_2(t)\rangle = \int_{-\infty}^{\infty} s_1(t)s_2^*(t)\,dt = \int_0^4 s_1(t)s_2^*(t)\,dt = \int_0^2 s_1(t)s_2^*(t)\,dt + \int_2^4 s_1(t)s_2^*(t)\,dt$$

$$= \int_0^2 (1)(2)\,dt + \int_2^4 (1)(0)\,dt = 2 \times 2 + 0 = 4.$$

c. Let $u(t) = s_2(t) - s_1(t)$.

i. Plot $u(t)$.



Similar to the integration for the inner product, we consider two time intervals: from 0 to 2 and from 2 to 4.

From $t = 0$ to $t = 2$, $s_2(t) - s_1(t) = 2 - 1 = 1$.

From $t = 2$ to $t = 4$, $s_2(t) - s_1(t) = 0 - 1 = -1$.

ii. Calculate the energy of $u(t)$

$E_u = \int_{-\infty}^{\infty} |u(t)|^2\,dt = \int_0^2 |1|^2\,dt + \int_2^4 |-1|^2\,dt = (2 \times 1) + (2 \times 1) = 4.$

iii. Calculate the inner product $\langle s_1(t), u(t)\rangle$

$$\langle s_1(t), u(t)\rangle = \int_{-\infty}^{\infty} s_1(t)u^*(t)\,dt = \int_0^2 s_1(t)u^*(t)\,dt + \int_2^4 s_1(t)u^*(t)\,dt$$

$$= \int_0^2 (1)(1)\,dt + \int_2^4 (1)(-1)\,dt = (2 \times 1) + \big(2 \times (-1)\big) = 0.$$

ECS 452: In-Class Exercise # 22

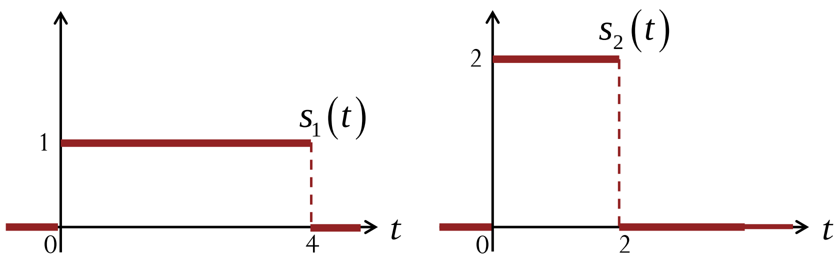
Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 21 / 4 / 2020			
Name	ID (last 3 digits)		

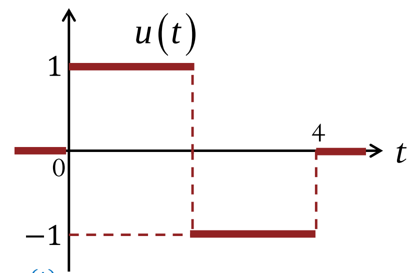
1. Continue from Exercise # 21. (You may use the results from the solution of Exercise # 21 as well.)

Consider two signals $s_1(t)$ and $s_2(t)$ shown below.



Let $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_{s_1}}} = \frac{1}{2}s_1(t)$ and

$\phi_2(t) = \frac{u(t)}{\sqrt{E_u}} = \frac{s_2(t)-s_1(t)}{2}$.



a. Show that $\phi_1(t)$ and $\phi_2(t)$ are orthonormal.

We have seen that for any energy signal $x(t)$, the normalized signal $\frac{x(t)}{\sqrt{E_x}}$ always has unit energy.

Therefore, there is no need to check that $\phi_1(t)$ and $\phi_2(t)$ **have unit energy**.

$$\langle \phi_1(t), \phi_2(t) \rangle = \langle \tfrac{1}{2}s_1(t), \tfrac{1}{2}u(t) \rangle = \tfrac{1}{2} \times \tfrac{1}{2} \times \langle s_1(t), u(t) \rangle = \tfrac{1}{2} \times \tfrac{1}{2} \times 0 = 0.$$

Property (2)

From Exec. 21, $\langle s_1, u \rangle = 0$.

➔ So, $\phi_1(t)$ and $\phi_2(t)$ are **orthogonal**.

$\phi_1(t)$ and $\phi_2(t)$ are **orthonormal.**

(See properties of inner products on the next page)

b. Calculate the following inner products:

$$\langle s_1, \phi_1 \rangle = \langle s_1, \tfrac{1}{2}s_1 \rangle = \tfrac{1}{2}\langle s_1, s_1 \rangle = \tfrac{1}{2}E_{s_1} = \tfrac{1}{2}(4) = 2.$$

Property (2)

From Exec. 21, $E_{s_1} = 4$.

$$\langle s_2, \phi_1 \rangle = \langle s_2, \tfrac{1}{2}s_1 \rangle = \tfrac{1}{2}\langle s_2, s_1 \rangle = \tfrac{1}{2}\langle s_1, s_2 \rangle = \tfrac{1}{2}(4) = 2.$$

Property (2)

Property (1)

From Exec. 21, $\langle s_1, s_2 \rangle = 4$.

$$\langle s_1, \phi_2 \rangle = \langle s_1, \tfrac{s_2 - s_1}{2} \rangle = \tfrac{1}{2}\langle s_1, u \rangle = \tfrac{1}{2} \times 0 = 0.$$

Property (2)

From Exec. 21, $\langle s_1, u \rangle = 0$.

$$\langle s_2, \phi_2 \rangle = \langle s_2, \tfrac{s_2 - s_1}{2} \rangle = \tfrac{1}{2}\langle s_2, s_2 - s_1 \rangle = \tfrac{1}{2}(\langle s_2, s_2 \rangle - \langle s_2, s_1 \rangle) = \tfrac{1}{2}(E_{s_2} - \langle s_1, s_2 \rangle) = \tfrac{1}{2}(8-4) = 2.$$

Property (2)

Property (4)

Property (1)

From Exec. 21, $\langle s_1, s_2 \rangle = 4$.

c. Suppose we use $\phi_1(t)$ and $\phi_2(t)$ as the new axes. Find the corresponding vectors $\vec{\mathbf{s}}^{(1)}$ and $\vec{\mathbf{s}}^{(2)}$ that represent $s_1(t)$ and $s_2(t)$ in the new coordinate system defined by $\phi_1(t)$ and $\phi_2(t)$.

$$\vec{\mathbf{s}}^{(1)} = \begin{pmatrix} \langle s_1(t), \phi_1(t) \rangle \\ \langle s_1(t), \phi_2(t) \rangle \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{s}}^{(2)} = \begin{pmatrix} \langle s_2(t), \phi_1(t) \rangle \\ \langle s_2(t), \phi_2(t) \rangle \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Many properties involving inner products follow from properties of integration.
Here are some useful properties:

(1) For two real-valued waveforms $x(t)$ and $y(t)$,

$$\langle y(t), x(t)\rangle = \int_{-\infty}^{\infty} y(t)x(t)\,dt = \int_{-\infty}^{\infty} x(t)y(t)\,dt = \langle x(t), y(t)\rangle.$$

(2) For two real-valued waveforms $x(t)$ and $y(t)$ and two real-valued constant a and b,

$$\langle ax(t), by(t)\rangle = \int_{-\infty}^{\infty} \big(ax(t)\big)\big(by(t)\big)\,dt = ab\int_{-\infty}^{\infty} \big(x(t)\big)\big(y(t)\big)\,dt = ab\langle x(t), y(t)\rangle.$$

(3) For a real-valued waveforms $x(t)$ and a real-valued constant c,

$$\|cx(t)\| = \sqrt{\langle cx(t), cx(t)\rangle} = \sqrt{c^2\langle x(t), x(t)\rangle} = |c|\sqrt{\langle x(t), x(t)\rangle} = |c|\|x(t)\|.$$

(4) For three real-valued waveforms $x(t)$, $y_1(t), y_2(t)$ and two real-valued constant c_1 and c_2,

$$\begin{aligned}\langle x(t), c_1y_1(t) + c_2y_2(t)\rangle &= \int_{-\infty}^{\infty} x(t)\big(c_1y_1(t) + c_2y_2(t)\big)\,dt \\ &= c_1\int_{-\infty}^{\infty} x(t)y_1(t)\,dt + c_2\int_{-\infty}^{\infty} x(t)y_2(t)\,dt \\ &= c_1\langle x(t), y_1(t)\rangle + c_2\langle x(t), y_2(t)\rangle.\end{aligned}$$

ECS 452: In-Class Exercise # 23

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 24 / 4 / 2020			
Name	ID (last 3 digits)		

A digital communication system transmits a stream of bits by mapping each block of three bits to one of the possible waveforms $s_1(t), s_2(t), \ldots, s_M(t)$. The waveform is then transmitted via a communication channel which corrupts the waveform by independently adding a white noise process $N(t)$ whose power spectral density is given by $S_N(f) \equiv 16$ across all frequency.

a. What is the value of M ?

The modulator uses blocks of three bits as its input. There are 2^3 possibilities for three-bit blocks. Each possible block should be mapped to a unique waveform So, there should be $M = 2^3 = 8$ possible waveforms.

b. Consider two orthonormal functions $\phi_1(t)$ and $\phi_2(t)$. Let $N_j = \langle N(t), \phi_j(t) \rangle$. Find

Because $N(t)$ is a white noise process, we can apply [7.26] from the lecture note. Here, $\frac{N_0}{2} = 16$.

i. $\mathbb{E}[N_1] = 0$ by [7.26.f.i].

ii. $\mathrm{Var}[N_1] = \mathbb{E}[N_1^2] - (\mathbb{E}[N_1])^2 = \frac{N_0}{2} - (0)^2 = 16.$

$\mathbb{E}[N_1] = 0$ by [7.26.f.i] or by part (i) above.

$\mathbb{E}[N_1^2] = \mathbb{E}[N_1 N_1] = \frac{N_0}{2}$ by [7.26.f.ii] with $i = j = 1$.

iii. $\sigma_{N_1} = \sqrt{\mathrm{Var}[N_1]} = \sqrt{16} = 4.$

$\mathrm{Var}[N_1] = 16$ by part (ii) above.

iv. $\mathbb{E}[N_1 N_2] = 0$ by [7.26.f.ii] with $i \neq j$.

ECS 452: In-Class Exercise # 24

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 28 / 4 / 2020			
Name	ID (last 3 digits)		

In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-2, 2\}$ with $p_1 = P[S = -2] = 0.6$ and $p_2 = P[S = 2] = 0.4$. The message is corrupted by an independent additive noise N. The detector observes $R = S + N$ at its input.

a. Suppose that the noise is a *discrete* random variable whose pmf is $p_N(n) = \begin{cases} 0.2, & n \in \{-2, 2\}, \\ 0.6, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$

i. List all possible values of R.

When $S = -2$, we have $R = S + N = -2 + N$. N can be $-2,0,2$; so, R can be $-4, -2, 0$.
When $S = 2$, we have $R = S + N = 2 + N$. N can be $-2,0,2$; so, R can be $0,2,4$.
Combining the two cases, R can be $-4, -2, 0, 2, 4$.

ii. Find the corresponding $\mathbf{Q}$ matrix. (Recall that the $\mathbf{Q}$ matrix contains $P[R = r | S = s]$.)

Recall that $P[R = r | S = s] = p_N(r - s)$.

We first find the values of $r - s$.

$s \backslash r$	-4	-2	0	2	4
-2	-2	0	2	4	6
2	-6	-4	-2	0	2

Then, we calculate $p_N(r - s)$ from the provided expression.

$s \backslash r$	-4	-2	0	2	4
-2	0.2	0.6	0.2	0	0
2	0	0	0.2	0.6	0.2

iii. Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

When the noise is a discrete RV, we follow the technique from [3.40] in CH3. Starting with the $\mathbf{Q}$ matrix, we scale its rows by the prior probabilities to get the $\mathbf{P}$ matrix.

$s \backslash r$	-4	-2	0	2	4
-2	0.2	0.6	0.2	0	0
2	0	0	0.2	0.6	0.2

-2 row $\times 0.6$; 2 row $\times 0.4$:

$s \backslash r$	-4	-2	0	2	4
-2	0.12	0.36	0.12	0	0
2	0	0	0.08	0.24	0.08

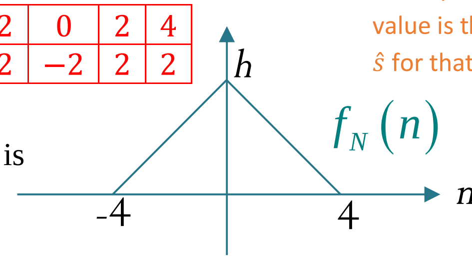
Then, we select the max. value in each column. The corresponding s value is the value of $\hat{s}$ for that r.

$$\hat{s}_{\text{MAP}}(r) = \begin{cases} -2, & r \in \{-4, -2, 0\}, \\ 2, & r \in \{2, 4\}. \end{cases}$$

or, in tabular form:

r	-4	-2	0	2	4
$\hat{s}_{\text{MAP}}(r)$	-2	-2	-2	2	2

b. Suppose that the noise is a *continuous* random variable whose pdf is



i. What is the value of h?

Area must be 1. Here, we have $\frac{1}{2} \times h \times 8 = 1$. Therefore, $h = \frac{1}{4}$.

ii. Suppose $R = 0$ is observed. Find the corresponding output of the MAP detector.

$$p_1 f_N(r - s^{(1)}) = 0.6 f_N(0 - (-2)) = 0.6 \times f_N(2) = 0.6 \times \frac{h}{2} = 0.3h = 0.075$$

$$p_2 f_N(r - s^{(2)}) = 0.4 f_N(0 - 2) = 0.4 \times f_N(-2) = 0.4 \times \frac{h}{2} = 0.2h = 0.050$$

Because $p_1 f_N(r - s^{(1)}) > p_2 f_N(r - s^{(2)})$ at $r = 0$, we conclude that $\hat{s}_{\text{MAP}}(0) = s^{(1)} = -2$.

ECS 452: In-Class Exercise # 26

Instructions

1. Working alone is always permitted. However, working in groups is also allowed if social distancing can be used (via, e.g., online group chat/call). For group work, **the group cannot be the same as any of your former group after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 5 / 5 / 2020			
Name	ID (last 3 digits)		

In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-2, 2\}$ with $p_1 = P[S = -2] = 0.6$ and $p_2 = P[S = 2] = 0.4$. The message is corrupted by an independent additive noise N. The detector observes $R = S + N$ at its input. Suppose $N \sim \mathcal{N}(0, 4)$.

1. Suppose the received symbol is $R = r$. Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

Let τ_{MAP} be the value of r at which $p_1 f_N(r - s^{(1)}) = p_2 f_N(r - s^{(2)})$.

Here, $p_1 = 0.6, p_2 = 0.4, s^{(1)} = -2, s^{(2)} = 2$, and $f_N(n) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{1}{2}\left(\frac{n}{\sigma_N}\right)^2}$ where $\sigma_N = \sqrt{4} = 2$. So,

$$0.6 \frac{1}{\sqrt{2\pi}2} e^{-\frac{1}{2}\left(\frac{r-(-2)}{2}\right)^2} = 0.4 \frac{1}{\sqrt{2\pi}2} e^{-\frac{1}{2}\left(\frac{r-2}{2}\right)^2}$$

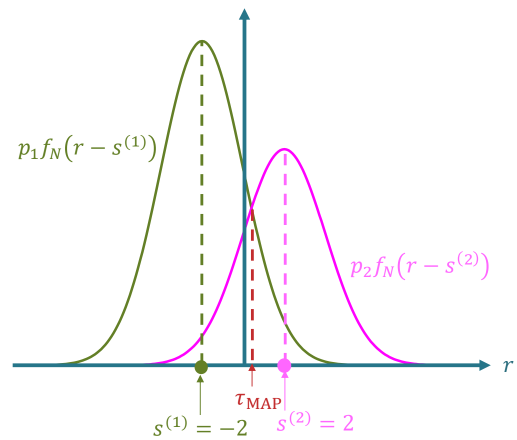
$$\frac{0.6}{0.4} = e^{-\frac{1}{2\times 2^2}\left((r-2)^2 - (r-(-2))^2\right)}$$

$$= e^{-\frac{1}{8}\left((r^2-4r+4)-(r^2+4r+4)\right)}$$

$$= e^{-\frac{1}{8}(-8r)} = e^r$$

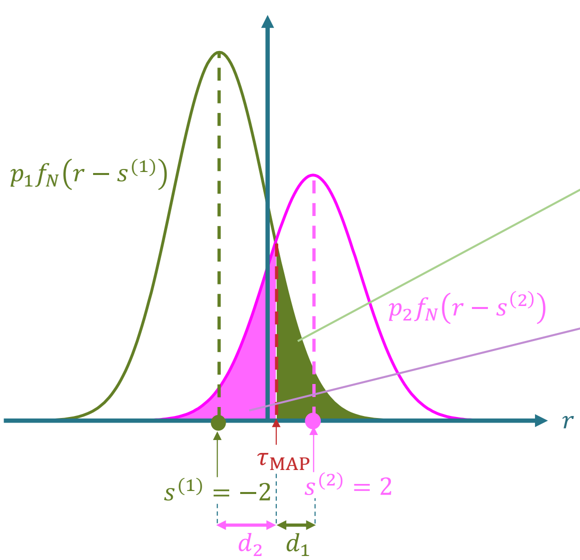
$$\frac{3}{2} = e^r$$

$$r = \ln\left(\frac{3}{2}\right) \approx 0.4055$$



$$\tau_{\text{MAP}} \approx 0.4055 \Rightarrow \hat{s}_{\text{MAP}}(r) = \begin{cases} s^{(1)}, & r < \tau_{\text{MAP}}, \\ s^{(2)}, & r \geq \tau_{\text{MAP}} \end{cases} \approx \begin{cases} -2, & r < 0.4055, \\ 2, & r \geq 0.4055. \end{cases}$$

2. Evaluate the corresponding error probability of the MAP detector. Your answer should be of the form $a_1 Q(b_1) + a_2 Q(b_2) + a_3 Q(b_3) + \cdots$ where the a_i and b_i are nonnegative constants.



From the lecture notes, we have seen that $P(\mathcal{E})$ is the sum of the highlighted areas.

$$\text{Area} = p_1 Q\left(\frac{d_1}{\sigma_N}\right) = p_1 Q\left(\frac{\tau_{\text{MAP}} - (-2)}{\sigma_N}\right)$$

$$\approx p_1 Q\left(\frac{0.4055 + 2}{2}\right) = 0.6Q(1.2027)$$

$$\text{Area} = p_2 Q\left(\frac{d_2}{\sigma_N}\right) = p_1 Q\left(\frac{2 - \tau_{\text{MAP}}}{\sigma_N}\right)$$

$$\approx p_2 Q\left(\frac{2 - 0.4055}{2}\right) = 0.4Q(0.7973)$$

$$P(\mathcal{E}) \approx 0.6Q(1.2027) + 0.4Q(0.7973)$$

$$\approx 0.1538$$

ECS 452: In-Class Exercise # 26

Instructions

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2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 5 / 5 / 2020			
Name	ID (last 3 digits)		

In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-2, 2\}$ with $p_1 = P[S = -2] = 0.6$ and $p_2 = P[S = 2] = 0.4$. The message is corrupted by an independent additive noise N. The detector observes $R = S + N$ at its input. Suppose $N \sim \mathcal{N}(0, 4)$.

1. Suppose the received symbol is $R = r$. Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

Let τ_{MAP} be the value of r at which $p_1 f_N(r - s^{(1)}) = p_2 f_N(r - s^{(2)})$.

Here, $p_1 = 0.6, p_2 = 0.4, s^{(1)} = -2, s^{(2)} = 2$, and $f_N(n) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{1}{2}\left(\frac{n}{\sigma_N}\right)^2}$ where $\sigma_N = \sqrt{4} = 2$. So,

$$0.6 \frac{1}{\sqrt{2\pi}2} e^{-\frac{1}{2}\left(\frac{r-(-2)}{2}\right)^2} = 0.4 \frac{1}{\sqrt{2\pi}2} e^{-\frac{1}{2}\left(\frac{r-2}{2}\right)^2}$$

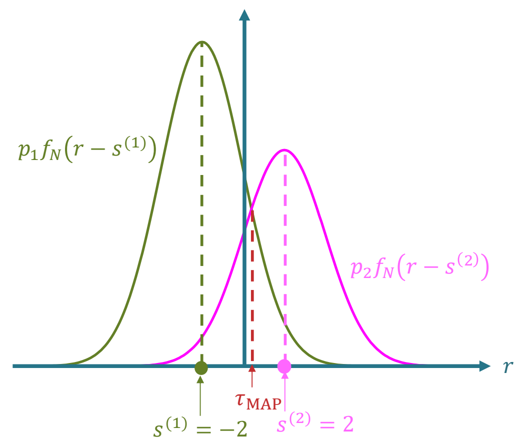
$$\frac{0.6}{0.4} = e^{-\frac{1}{2\times 2^2}\left((r-2)^2 - (r-(-2))^2\right)}$$

$$= e^{-\frac{1}{8}\left((r^2-4r+4)-(r^2+4r+4)\right)}$$

$$= e^{-\frac{1}{8}(-8r)} = e^r$$

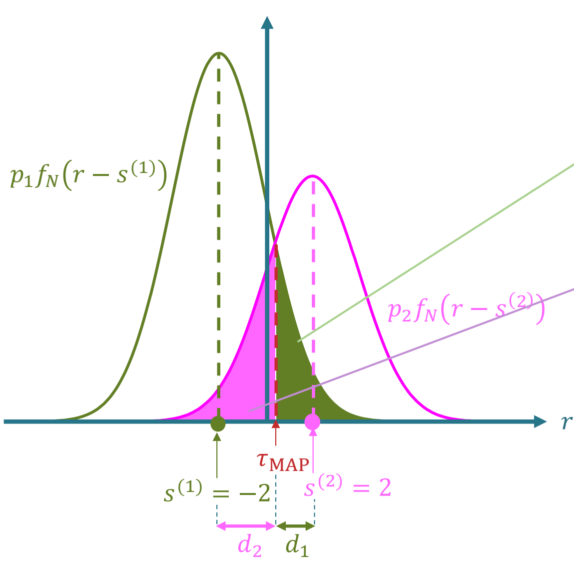
$$\frac{3}{2} = e^r$$

$$r = \ln\left(\frac{3}{2}\right) \approx 0.4055$$



$$\tau_{\text{MAP}} \approx 0.4055 \Rightarrow \hat{s}_{\text{MAP}}(r) = \begin{cases} s^{(1)}, & r < \tau_{\text{MAP}}, \\ s^{(2)}, & r \ge \tau_{\text{MAP}} \end{cases} \approx \begin{cases} -2, & r < 0.4055, \\ 2, & r \ge 0.4055. \end{cases}$$

2. Evaluate the corresponding error probability of the MAP detector. Your answer should be of the form $a_1 Q(b_1) + a_2 Q(b_2) + a_3 Q(b_3) + \cdots$ where the a_i and b_i are nonnegative constants.



$P(\mathcal{E})$ is the sum of the highlighted areas.

$$\text{Area}_1 = p_1 Q\left(\frac{d_1}{\sigma_N}\right) = p_1 Q\left(\frac{\tau_{\text{MAP}} - (-2)}{\sigma_N}\right)$$

$$\approx p_1 Q\left(\frac{0.4055 + 2}{2}\right) = 0.6Q(1.2027)$$

$$\text{Area}_2 = p_2 Q\left(\frac{d_2}{\sigma_N}\right) = p_1 Q\left(\frac{2 - \tau_{\text{MAP}}}{\sigma_N}\right)$$

$$\approx p_2 Q\left(\frac{2 - 0.4055}{2}\right) = 0.4Q(0.7973)$$

$$P(\mathcal{E}) = \text{Area}_1 + \text{Area}_2$$

$$\approx 0.6Q(1.2027) + 0.4Q(0.7973)$$

$$\approx 0.1538$$