ECS 452: In-Class Exercise # 5 Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 31 / 1 / 2020			
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1. In each part below, we consider a random variable X which has five possible values. The probability for each possible value is listed in the provided table. Calculate the corresponding entropy value.

a.

x	a	e	ℓ	n	r
$p(x)$	0.25	0.25	0.125	0.25	0.125

$$H(X) = -\sum_x p(x)\log_2 p(x) = -(3\times 0.25\times\log_2 0.25 + 2\times 0.125\times\log_2 0.125)$$

$$= -\left(3\times\frac{1}{4}\times\log_2\frac{1}{4} + 2\times\frac{1}{8}\times\log_2\frac{1}{8}\right) = 3\times\frac{1}{4}\times 2 + 2\times\frac{1}{8}\times 3 = \frac{6}{4}+\frac{3}{4} = \frac{9}{4} = 2.25\text{ [bits]}$$

b.

x	a	e	ℓ	n	r
$p(x)$	0.1	0.2	0.2	0.2	0.3

$$H(X) = -\sum_x p(x)\log_2 p(x) = -(0.1\times\log_2 0.1 + 3\times 0.2\times\log_2 0.2 + 0.3\times\log_2 0.3)$$

$$\approx 0.1\times 3.3219 + 3\times 0.2\times 2.3219 + 0.3\times 1.7370 \approx 0.3322 + 3\times 0.4644 + 0.5211$$

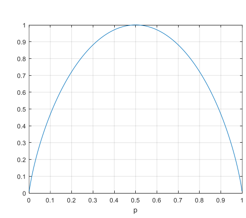
$$\approx 2.2464\text{ [bits]}$$

2. [ENRPr] In each row of the following table, compare the quantity in the first column with the one in the third column by writing ">", "=", or "<" in the second column. Watch out for approximation error.

(a)	$H(\underline{\mathbf{p}}) \approx 0.9710$ when $\underline{\mathbf{p}} = [0.4, 0.6]$	$>$	$H(\underline{\mathbf{p}}) \approx 0.8813$ when $\underline{\mathbf{p}} = [0.3, 0.7]$
(b)	$H(\underline{\mathbf{p}}) \approx 0.9710$ when $\underline{\mathbf{p}} = [0.4, 0.6]$	$<$	$H(\underline{\mathbf{p}}) \approx 1.2955$ when $\underline{\mathbf{p}} = [0.1, 0.3, 0.6]$
(c)	$H(\underline{\mathbf{p}}) \approx 1.8464$ when $\underline{\mathbf{p}} = [0.1, 0.2, 0.3, 0.4]$	$=$	$H(\underline{\mathbf{p}}) \approx 1.8464$ when $\underline{\mathbf{p}} = [0.1, 0.3, 0.2, 0.4]$

First, note that the arguments of the entropy function are vectors. The vectors themselves represent probability mass functions. (Note that the sum of the elements in each vector is one.) Given a probability vector $\underline{\mathbf{p}}$,

$$H(\underline{\mathbf{p}}) \equiv -\sum_i p_i \log_2 p_i.$$



We can try to calculate the entropy values directly to compare them. This is done in the table above.

Alternatively, we can avoid directly computing the values by the following observation:

a) When $\underline{\mathbf{p}} = [p_1, p_2]$, we have

$$H(\underline{\mathbf{p}}) \equiv -p_1\log_2 p_1 - p_2\log_2 p_2 = -p_1\log_2 p_1 - (1-p_1)\log_2(1-p_1).$$

This is the same as the binary entropy function $H(p)$ evaluated at $p = p_1$. In class, we have seen the plot of $H(p)$ as a function of p. When the value of p is closer to 0.5, the entropy value is larger. Here, 0.4 is closer to 0.5 than 0.3 is.

b) We are comparing $-0.4\log_2 0.4 - 0.6\log_2 0.6$ and $-0.1\log_2 0.1 - 0.3\log_2 0.3 - 0.6\log_2 0.6$.
Both quantities have the term "$-0.6\log_2 0.6$"; so, we can ignore this term.
Now, note that $0.4 = 0.1 + 0.3$. So,

$$-0.4\log_2 0.4 = -(0.1+0.3)\log_2 0.4 = -0.1\log_2 0.4 - 0.3\log_2 0.4. \quad ❶$$

Because $\log_2(\cdot)$ is an increasing function, we know that $\log_2 0.4$ is greater than both $\log_2 0.1$ and $\log_2 0.3$.
With the negative signs, $-\log_2 0.4$ is less than both $-\log_2 0.1$ and $-\log_2 0.3$.
Therefore, continued from ❶, we have

$$-0.4\log_2 0.4 = -0.1\log_2 0.4 - 0.3\log_2 0.4 < -0.1\log_2 0.1 - 0.3\log_2 0.3.$$

c) The elements in vectors $\underline{\mathbf{p}}$ are the same on both sides except that the ordering is different. By the commutative property of addition, the resulting entropy values should be the same.