ECS 452: In-Class Exercise # 1

Instructions

1. Separate into groups of no more than three persons. Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **24** / **01** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider two codes (for source coding) below. The left column is for Code A. The right column is for Code B. The first row defines these codes via their codebooks.

Caution: the code alphabet is NOT the collection of all possible codewords.

<table>
<tr>
<td>Codebook for Code A
<table>
<tr><td>x</td><td>E</td><td>L</td><td>M</td><td>N</td><td>O</td></tr>
<tr><td>$c(x)$</td><td>101</td><td>110</td><td>111</td><td>011</td><td>100</td></tr>
</table>
</td>
<td>Codebook for Code B
<table>
<tr><td>x</td><td>E</td><td>L</td><td>M</td><td>N</td><td>O</td></tr>
<tr><td>$c(x)$</td><td>0</td><td>100</td><td>1010</td><td>1011</td><td>11</td></tr>
</table>
The code alphabet is the collection of all possible code symbols that we can used to construct the codewords.
</td>
</tr>
<tr>
<td>The <u>source</u> alphabet for Code A is
The source alphabet is the collection of all possible source symbols. Therefore, it can be easily extracted from the codebook: {E, L, M, N, O}</td>
<td>The <u>code</u> alphabet for Code B is
Here, we see that the symbols used for each codeword are 0 and 1. Therefore, the code alphabet is {0,1}</td>
</tr>
<tr>
<td>Use code A to encode the source string “NONE”
011|100|011|101</td>
<td>Use code B to encode the source string “NONE”
1011|11|1011|0</td>
</tr>
<tr>
<td>Is Code A nonsingular? A code is nonsingular if different source symbols are mapped to different codewords.
All five codewords in the codebook are different. Therefore, yes, the code is nonsingular.</td>
<td>Is Code B nonsingular?
All five codewords in the codebook are different. Therefore, yes, the code is nonsingular.</td>
</tr>
<tr>
<td>The string 110|101|111|100|011 is from encoding by Code A.
L E M O N
Decode it.
Decode string: LEMON</td>
<td>The string 1010|0|100|11|1011 is from encoding by Code B. Decode it.
M E L O N
Decode string: MELON</td>
</tr>
</table>

2. Suppose we don’t use letter space and word space in Morse code. Consider the following encoded string: ●●● ▬ ▬ ▬ ●●●
Note that “SOS” and “EEATB” are two possible interpretations. Find four additional interpretations.

Indicate how the codewords are separated by “/”	Decoded message
●●●/▬ ▬ ▬/●●●	SOS
●/●/● ▬/▬/▬ ●●●	EEATB
●●● ▬ ▬/▬ ●●●	**3B**
●●● ▬/▬ ▬ ●●●	**V7**
●●/● ▬ ▬ ▬/●●●	**IJS**
●●●/▬ ▬ ▬ ●●/●	**S8E**

A	● ▬	U	● ● ▬
B	▬ ● ● ●	V	● ● ● ▬
C	▬ ● ▬ ●	W	● ▬ ▬
D	▬ ● ●	X	▬ ● ● ▬
E	●	Y	▬ ● ▬ ▬
F	● ● ▬ ●	Z	▬ ▬ ● ●
G	▬ ▬ ●		
H	● ● ● ●		
I	● ●		
J	● ▬ ▬ ▬		
K	▬ ● ▬	1	● ▬ ▬ ▬ ▬
L	● ▬ ● ●	2	● ● ▬ ▬ ▬
M	▬ ▬	3	● ● ● ▬ ▬
N	▬ ●	4	● ● ● ● ▬
O	▬ ▬ ▬	5	● ● ● ● ●
P	● ▬ ▬ ●	6	▬ ● ● ● ●
Q	▬ ▬ ● ▬	7	▬ ▬ ● ● ●
R	● ▬ ●	8	▬ ▬ ▬ ● ●
S	● ● ●	9	▬ ▬ ▬ ▬ ●
T	▬	0	▬ ▬ ▬ ▬ ▬

There are other solutions as well.

ECS 452: In-Class Exercise # 2

Instructions

1. Separate into groups of no more than three persons. The group cannot be the same as any of your former groups. Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **25**/ **01** / 2019

Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider a DMS whose source alphabet is {E,L,M,N,O}.
   The probabilities for these five symbols are shown in the table below:

x	E	L	M	N	O
$p(x)$	0.1	0.1	0.2	0.2	0.4

Consider two codes (for source coding) below.
The left column is for Code A. The right column is for Code B.
The first row defines these codes via their codebooks.

Codebook for Code A

x	E	L	M	N	O
$c(x)$	101	110	111	011	100

Is Code A prefix-free?

Yes, no codeword is a prefix of another codeword.

Observation: Any fixed-length non-singular codes are also prefix-free.

Suppose the DMS above is encoded by Code A.
Find the expected codeword length.

The length of all code word is 3.
Therefore,

$\mathbb{E}[\ell(X)]$ = **3 bits per source symbol**

Codebook for Code B

x	E	L	M	N	O
$c(x)$	0	100	1010	1011	11
$\ell(x)$	1	3	4	4	2

Is Code B prefix-free?

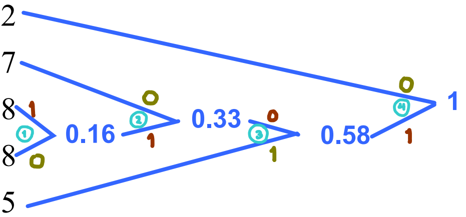
Yes, no codeword is a prefix of another codeword.

Remark: Some codewords have other codewords as their suffixes. However, we only consider prefix, not suffix.

Suppose the DMS above is encoded by Code B.
Find the expected codeword length.

$$\mathbb{E}[\ell(X)] = 0.1(1+3) + 0.2(4+4) + 0.4 \times 2$$
$$= 0.4 + 1.6 + 0.8$$
$$= 2.8 \text{ bits per source symbol}$$

2. Consider a random variable X which has five possible values. Their probabilities are shown in the table below.

x	$p_X(x)$	$c(x)$	$\ell(x)$
E	0.42  **The tree can be contruct by following Huffman's recipe.** **The grouping orders are indicated by circled numbers.** **The code symbols on each branch are forced by having to make 1011 the codeword for M.**	**0**	**1**
L	0.17	**100**	**3**
M	0.08	1011	**4**
N	0.08	**1010**	**4**
O	0.25	**11**	**2**

a. Find a binary Huffman code (without extension) for this random variable.
   Put the values of the codewords and the codeword lengths in the table above.
   Note that the codeword for the source symbol "M" is required to be 1011.

b. Find the expected codeword length when Huffman coding is used (without extension).

= 0.42x1 + 0.17x3 + (0.08+0.08)x4 + 2x0.25
= 0.42 + 0.51 + 0.64 + 0.50
= 2.07 [bits per source symbol]

ECS 452: In-Class Exercise # 3

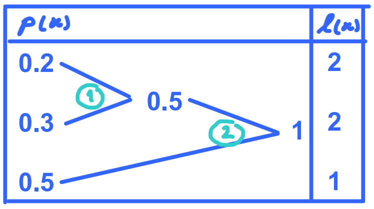
Date: **08** / **02** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

Instructions

1. Separate into groups of no more than three persons. The group cannot be the same as any of your former groups. Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

1. A discrete memoryless source emits three possible messages Yes, No, and OK with probabilities 0.2 and 0.3, and 0.5, respectively.
   a. Find the expected codeword length when Huffman binary code is used without extension.

The grouping orders are indicated by circled numbers.

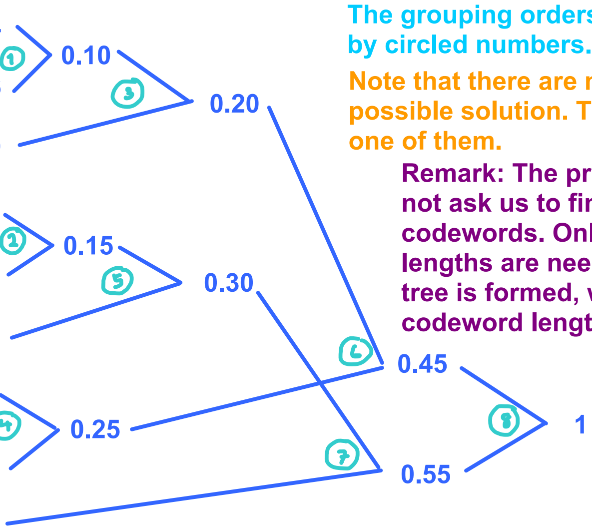


0.2x2 + 0.3x2 + 0.5x1 = 1.5 bits per source symbol

Remark: The problem does not ask us to find the codewords. Only the codeword lengths are needed. Once the tree is formed, we can read the codeword lengths directly.

   b. Find the codeword lengths when Huffman binary code with second-order extension is used to encode this source. Put the values of the corresponding probabilities and the codeword lengths in the table below. (Note that, for brevity, we use Y,N,K to represent Yes, No, and OK, respectively.)

x_1x_2	$p_{X_1,X_2}(x_1,x_2)$	$\ell(x_1,x_2)$
YY	**0.2x0.2 = 0.04**	**4**
YN	**0.2x0.3 = 0.06**	**4**
YK	**0.2x0.5 = 0.10**	**3**
NY	**0.3x0.2 = 0.06**	**4**
NN	**0.3x0.3 = 0.09**	**4**
NK	**0.3x0.5 = 0.15**	**3**
KY	**0.5x0.2 = 0.10**	**3**
KN	**0.5x0.3 = 0.15**	**3**
KK	**0.5x0.5 = 0.25**	**2**



The grouping orders are indicated by circled numbers.

Note that there are many possible solution. This is just one of them.

Remark: The problem does not ask us to find the codewords. Only the codeword lengths are needed. Once the tree is formed, we can read the codeword lengths directly.

Note that even when the Huffman'recipe is followed strictly, there are many possible solutions. For example, at Step 3, there are three choices of 0.1 that we can choose.

   c. Find L_2.
(This is the expected codeword length **per source symbol** of the Huffman binary code for the second-order extension of this source.)

(0.04+0.06+0.06+0.09)x4 + (0.10+0.15+0.10+0.15)x3 + 0.25x2
= 0.25x4 + 0.5x3 + 0.5 = 3 bits per two source symbols.

$L_2 = \frac{3}{2}$ = **1.5 bits per source symbol**

Same as part (a). So, the second-order extension does not help in this case.

ECS 452: In-Class Exercise # 4

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **08** / **02** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Write each of the following quantities in the form X.XXX (possibly with the help of your calculator).

   a. $-\log_2(1/\underbrace{128}_{2^7}) = -\log_2\left(\frac{1}{2^7}\right) = -\log_2(2^{-7}) = -(-7)\underbrace{\log_2 2}_{1} =$ **7.000**

   b. $-\log_2(0.6) \approx$ **0.737**

   Method 1

   $$-\log_2 a = \frac{-\log_e a}{\log_e 2} = -\frac{\ln(0.6)}{\ln(2)} \approx -\frac{-0.5108}{0.6931} \approx 0.7370$$

   Method 2

   $$-\log_2 a = \frac{-\log_{10} a}{\log_{10} 2} = -\frac{\log_{10}(0.6)}{\log_{10}(2)} \approx -\frac{-0.2218}{0.3010} \approx 0.7369$$

   c. $\underbrace{-(0.4)\underbrace{\log_2(0.4)}_{-1.3219}}_{0.5288}\ \underbrace{-(0.6)\underbrace{\log_2(0.6)}_{-0.7370}}_{0.4422} \approx$ **0.971**

2. In each part below, we consider a random variable *X* which has five possible values. The probability for each possible value is listed in the provided table. Calculate the corresponding entropy value.

   a.

x	E	L	M	N	O
$p(x)$	0.25	0.25	0.25	0.125	0.125

$$H(X) = -\sum_{x} p(x)\log_2 p(x) = -3\times\frac{1}{4}\log_2\frac{1}{4} - 2\times\frac{1}{8}\log_2\frac{1}{8}$$

$$= -3\times\frac{1}{4}\times(-2) - \cancel{2}\times\frac{1}{\underset{4}{\cancel{8}}}\times(-3) = \frac{9}{4} = 2.25 \quad \text{[bits]}$$

   b.

x	E	L	M	N	O
$p(x)$	0.1	0.1	0.2	0.2	0.4

$$H(X) = -2\times 0.1\times\underbrace{\log_2 0.1}_{-3.3219} - 2\times 0.2\underbrace{\log_2 0.2}_{-2.3219} - 0.4\underbrace{\log_2 0.4}_{-1.3219} \approx 2.1219 \text{ bits}$$

   c.

x	E	L	M	N	O
$p(x)$	0.42	0.17	0.08	0.08	0.25

$$H(X) = -0.42\underbrace{\log_2 0.42}_{-1.2515} - 0.17\underbrace{\log_2 0.17}_{-2.5564} - 2\times 0.08\underbrace{\log_2 0.08}_{-3.6439} - 0.25\underbrace{\log_2 0.25}_{-2} \approx 2.0433 \text{ bits}$$

ECS 452: In-Class Exercise # 5

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **14** / **02** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. No need to provide any explanation for this question.
Consider a DMC whose samples of input and output are provided below

```
x: 1 1 1 0 1 0 1 0 1 1 0 1 1 0 0
y: 1 1 1 0 1 0 1 1 0 1 1 0 1 1 0
```

Estimate the following quantities:

a. $\mathcal{X} = \{0, 1\}$ channel input alphabet = support of X

b. $P[X = 0] = \frac{6}{15} = \frac{2}{5} = 0.4$

c. $p(1) \equiv P[X=1] = 1 - P[X=0] = 1 - \frac{2}{5} = \frac{3}{5} = 0.6$

d. $p_Y(0) \equiv P[Y=0] = \frac{5}{15} = \frac{1}{3}$

e. $\underline{\mathbf{p}} \equiv [p(0) \;\; p(1)] = [\frac{2}{5} \;\; \frac{3}{5}] = [0.4 \;\; 0.6]$

f. $q(1) = P[Y=1] = 1 - P[Y=0] = 1 - \frac{1}{3} = \frac{2}{3}$

g. $P[Y = 0 | X = 0] = \frac{3}{6} = \frac{1}{2} = 0.5$

h. $p_{Y|X}(1|0) \equiv P[Y=1|X=0]$
$= 1 - P[Y=0|X=0] = 1 - 0.5 = 0.5$

i. $Q(0|1) \equiv P[Y=0|X=1] = \frac{2}{9}$

j. $Q(1|1) \equiv P[Y=1|X=1]$
$= 1 - P[Y=0|X=1] = 1 - \frac{2}{9} = \frac{7}{9}$

k. Matrix **Q**

$$\begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.5 & 0.5 \\ 1 & 2/9 & 7/9 \end{array}$$

l. $P[X = 0, Y = 0] = \frac{3}{15} = \frac{1}{5} = 0.2$

Alternative method:

$$P[\underbrace{X=0}_{A}, \underbrace{Y=0}_{B}] = P(A \cap B) = P(A)P(B|A)$$

$$= P[X=0]\,P[Y=0|X=0]$$

$$= \underbrace{0.4}_{\text{from part (b)}} \times \underbrace{0.5}_{\text{from part (g)}} = 0.2$$

ECS 452: In-Class Exercise # 6

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **15** / **02** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider a DMC whose transition matrix $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{array}{c|cccc} x\backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.3 & 0.4 & 0.2 & 0.1 \\ 2 & 0.2 & 0.5 & 0.1 & 0.2 \\ 3 & 0.1 & 0.3 & 0.3 & 0.3 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.5} \\ \xrightarrow{\times 0.1} \end{array} \quad \begin{array}{c|cccc} x\backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.12 & 0.16 & 0.08 & 0.04 \\ 2 & 0.10 & 0.25 & 0.05 & 0.10 \\ 3 & 0.01 & 0.03 & 0.03 & 0.03 \\ & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow \\ & [0.23 & 0.44 & 0.16 & 0.17] \end{array} = \mathbf{P}$$

Suppose the input probability vector is $\underline{\mathbf{p}} = [0.4 \quad 0.5 \quad 0.1]$.

a) Find the joint pmf matrix $\mathbf{P}$. Put your answer next to the $\mathbf{Q}$ matrix above.

b) Find the output probability vector $\underline{\mathbf{q}}$.

$[0.23 \quad 0.44 \quad 0.16 \quad 0.17]$

c) Suppose the naïve decoder is used. Find the corresponding $P(\mathcal{E})$.

$\hat{x} = y$

$$\mathbf{P} = \begin{array}{c|cccc} x\backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & \boxed{0.12} & 0.16 & 0.08 & 0.04 \\ 2 & 0.10 & \boxed{0.25} & 0.05 & 0.10 \\ 3 & 0.01 & 0.03 & \boxed{0.03} & 0.03 \end{array}$$

$P(\mathcal{C}) = 0.12 + 0.25 + 0.03$

$= 0.40$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.40 = 0.60$

d) Suppose the following decoder is used. Find the corresponding $P(\mathcal{E})$.

y	$\hat{x}(y)$
1	3
2	1
3	1
4	3

$$\mathbf{P} = \begin{array}{c|cccc} \hat{x} & 3 & 1 & 1 & 3 \\ x\backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.12 & \boxed{0.16} & \boxed{0.08} & 0.04 \\ 2 & 0.10 & 0.25 & 0.05 & 0.10 \\ 3 & \boxed{0.01} & 0.03 & 0.03 & \boxed{0.03} \end{array}$$

$P(\mathcal{C}) = 0.01 + 0.16 + 0.08 + 0.03$

$= 0.28$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.28$

$= 0.72$

e) Suppose the decoder is $\hat{x}(y) = 2.5 - |y - 2.5|$

Find the corresponding $P(\mathcal{E})$.

y	$y-2.5$	$2.5-\lvert y-2.5\rvert$
1	-1.5	1
2	-0.5	2
3	0.5	2
4	1.5	1

$$\mathbf{P} = \begin{array}{c|cccc} x\backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & \boxed{0.12} & 0.16 & 0.08 & \boxed{0.04} \\ 2 & 0.10 & \boxed{0.25} & \boxed{0.05} & 0.10 \\ 3 & 0.01 & 0.03 & 0.03 & 0.03 \end{array}$$

$P(\mathcal{C}) = 0.12 + 0.25 + 0.05 + 0.04$

$= 0.46$

$P(\mathcal{E}) = 1 - 0.46 = 0.54$

ECS 452: In-Class Exercise # 7

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **21** / **02** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider a DMC whose transition matrix $\mathbf{Q}$ and joint pmf matrix $\mathbf{P}$ are given below.

$$\mathbf{Q} = \begin{array}{c|cccc} x \setminus y & 1 & 2 & 3 & 4 \\ \hline 1 & \underline{0.3} & 0.4 & 0.2 & 0.1 \\ 2 & 0.2 & \underline{0.5} & 0.1 & 0.2 \\ 3 & 0.1 & 0.3 & \underline{0.3} & \underline{0.3} \end{array} \qquad \mathbf{P} = \begin{array}{c|cccc} x \setminus y & 1 & 2 & 3 & 4 \\ \hline 1 & \underline{\enclose{circle}{0.12}} & 0.16 & \enclose{circle}{0.08} & 0.04 \\ 2 & 0.10 & \underline{\enclose{circle}{0.25}} & 0.05 & \enclose{circle}{0.10} \\ 3 & 0.01 & 0.03 & \underline{0.03} & \underline{0.03} \end{array}$$

a) Find the MAP detector. Put your answer in the decoding table below. Also find the corresponding error probability.

y	$\hat{x}_{\text{MAP}}(y)$
1	**1**
2	**2**
3	**1**
4	**2**

$P(\mathcal{C}) = 0.12+0.25+0.08+0.10 = 0.55$

$P(\mathcal{E}) = 1-P(\mathcal{C}) = 1-0.55 = 0.45$

b) Find the ML detector. Put your answer in the decoding table below. Also find the corresponding error probability.

y	$\hat{x}_{\text{ML}}(y)$
1	**1**
2	**2**
3	**3**
4	**3**

$P(\mathcal{C}) = 0.12+0.25+0.03+0.03 = 0.43$

$P(\mathcal{E}) = 1-P(\mathcal{C}) = 1-0.43 = 0.57$

c) Find the pmf $p(x)$ of the channel input X.

Recall that to get the P matrix from the Q matrix, we multiply each row of the Q matrix by the corresponding p(x). So, to get p(x), we simply divides each row of the P matrix by the corresponding row in the Q matrix. (In fact, only one representative from each row is enough.)

first column of the P matrix

0.12/0.3 = 0.4
0.10/0.2 = 0.5
0.01/0.1 = 0.1

first column of the Q matrix

$$p(x) = \begin{cases} 0.4, & x = 1, \\ 0.5, & x = 2, \\ 0.1, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

Alternatively, recall, from ECS 315, that once the P matrix is there, one can get the pmf of X by summing along each row of the P matrix.

ECS 452: In-Class Exercise # 8

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **28** / **02** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider two random variables X and Y whose joint pmf matrix is given by

$$Q = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{matrix} \xleftarrow{\times 4} \\ \xleftarrow{\times 4} \\ \xleftarrow{\times 2} \end{matrix} \quad \mathbf{P} = \begin{array}{c|cccc|l} x\backslash y & 1 & 2 & 3 & 4 & \\ \hline 1 & 1/8 & 0 & 1/8 & 0 & \xrightarrow{\Sigma} 1/4 \\ 2 & 1/8 & 1/8 & 0 & 0 & \xrightarrow{\Sigma} 1/4 \\ 3 & 0 & 1/8 & 1/8 & 1/4 & \xrightarrow{\Sigma} 1/2 \\ \hline & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \\ & 1/4 & 1/4 & 1/4 & 1/4 & \end{array}$$

1/8 0 1/8 0
1/8 1/8 0 0
0 1/8 1/8 1/4

Calculate the following quantities.

a. $H(X,Y) \equiv -\sum_{(x,y)} p(x,y)\log_2 p(x,y) = -6\times\frac{1}{8}\log_2\frac{1}{8} - \frac{1}{4}\log_2\frac{1}{4}$

$= -\frac{3}{4}(-3) - \frac{1}{4}(-2) = \frac{9+2}{4} = \frac{11}{4} = 2.75$ **[bits per pair]**

pair of symbols (X,Y)

b. $H(X)$ **First, we find p(x) by summing along each row of the P matrix.**

$= -\sum_{x} p(x)\log_2 p(x) = -2\times\frac{1}{4}\log_2\frac{1}{4} - \frac{1}{2}\log_2\frac{1}{2} = -\frac{1}{2}(-2) - \frac{1}{2}(-1) = \frac{2+1}{2} = \frac{3}{2} = 1.5$

[bits per symbol]

c. $H(Y)$ **First, we find q(y) by summing along each column of the P matrix.**
We then know that Y is a uniform RV with four equally-likely possibilities.

Therefore, $H(Y) = \log_2 4 = 2$ **[bits per symbol]**

Alternatively, $H(Y) = -\sum_{y} q(y)\log_2 q(y)$
$= -\cancel{4}\times\frac{1}{\cancel{4}}\log_2\frac{1}{4}$
$= -\log_2 2^{-2} = 2$

d. $H(Y|X)$

$= H(X,Y) - H(X) = 2.75 - 1.5 = 1.25 = \frac{5}{4}$ **[bits per symbol]**

Note that this is calculating the (average) amount of randomness in Y (but given that we know the value of X). So the unit is per Y symbol.

e. **Q** matrix ← **can be found by scaling each row of the P matrix by** $\frac{1}{p(x)}$

$$\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{matrix} \xleftarrow{\times 4} \\ \xleftarrow{\times 4} \\ \xleftarrow{\times 2} \end{matrix} \begin{bmatrix} 1/8 & 0 & 1/8 & 0 \\ 1/8 & 1/8 & 0 & 0 \\ 0 & 1/8 & 1/8 & 1/4 \end{bmatrix}$$

x	$p(x)$	$1/p(x)$
1	1/4	4
2	1/4	4
3	1/2	2

f. $\underbrace{H(Y|X=3)}$

↳ **We use the "x = 3" row of the Q matrix to calculate this conditional entropy.**

$= -2\times\frac{1}{4}\log_2\frac{1}{4} - \frac{1}{2}\log_2\frac{1}{2} = -\frac{1}{2}(-2) - \frac{1}{2}(-1) = \frac{3}{2} = 1.5$

[bits per symbol]

We can also find $H(Y|X=1) = H(Y|X=2) = 1$.

Therefore, $H(Y|X) = \sum_{x} p(x) H(Y|x)$

$= \frac{1}{4}\times 1 + \frac{1}{4}\times 1 + \frac{1}{2}\times\frac{3}{2} = \frac{5}{4}$

same as what we got in part (d).

ECS 452: In-Class Exercise # 9

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **07** / **03** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider two random variables X and Y whose joint pmf matrix is given by $\mathbf{P}=\begin{bmatrix}0.2 & 0.2\\ 0.2 & 0.4\end{bmatrix}$. Find $I(X;Y)$.

We use the formula $I(X;Y)=H(X)+H(Y)-H(X,Y)$.

$H(X,Y)$ can be found directly from the elements in the P matrix:

$$H(X,Y) = -0.4\log_2 0.4 - 3\times 0.2\log_2 0.2 \approx 1.9219$$

$H(X)$ and $H(Y)$ can be found by first finding $p(x)$ and $q(y)$ from the P matrix:

$$P=\begin{bmatrix}0.2 & 0.2\\ 0.2 & 0.4\end{bmatrix}\begin{matrix}\rightarrow 0.4\\ \rightarrow 0.6\end{matrix}$$

$$\begin{matrix}\downarrow & \downarrow\\ 0.4 & 0.6\end{matrix}$$

$H(X) = -0.4\log_2 0.4 - 0.6\log_2 0.6 \approx 0.9710$

$H(Y) = 0.9710$

X and Y are identically distributed. So, they have the same entropy.

So, $I(X;Y) \approx 2\times 0.9710 - 1.9219$

$= 0.0200$

Note: when the answer here is small, it is important that you go back and make sure that you keep enough decimal paces in your calculation.

Note: You will get 0.0201 if you round to four decimal places too early.

2. Consider two random variables X and Y whose $\mathbf{p}=[0.6, 0.4]$ and $\mathbf{Q}=\begin{bmatrix}0.4 & 0.6\\ 0.7 & 0.3\end{bmatrix}$. Find $I(X;Y)$.

First, we find the P matrix by scaling each row of the Q matrix by the corresponding $p(x)$.

$$\begin{bmatrix}0.4 & 0.6\\ 0.7 & 0.3\end{bmatrix}\begin{matrix}\xrightarrow{\times 0.6}\\ \xrightarrow{\times 0.4}\end{matrix}\begin{bmatrix}0.24 & 0.36\\ 0.28 & 0.12\end{bmatrix} = P$$

$$\begin{matrix}\downarrow\Sigma & \downarrow\Sigma\end{matrix}$$

$$q = [0.52 \quad 0.48]$$

Then, we follow the same entropy calculations as in question (1).

$$H(X,Y) = -0.24\log_2 0.24 - 0.28\log_2 0.28 - 0.36\log_2 0.36 - 0.12\log_2 0.12$$

$$\approx 1.9060$$

$H(X) = -0.6\log_2 0.6 - 0.4\log_2 0.4 \approx 0.9710$

$H(Y) = -0.52\log_2 0.52 - 0.48\log_2 0.48 \approx 0.9988$

Alternatively

$H(Y|X=x_1) = H([0.4 \ 0.6]) \approx 0.9710$

$H(Y|X=x_2) = H([0.7 \ 0.3]) \approx 0.8813$

$H(Y|X) = \sum_x p(x)H(Y|x) \approx 0.6\times 0.9710 + 0.4\times 0.8813 \approx 0.9351$

$I(X;Y) = H(Y) - H(Y|X) \approx 0.9988 - 0.9351 \approx 0.0638$

$I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.9710 + 0.9988 - 1.9060 \approx 0.0638$

Note: when the answer here is small, it is important that you go back and make sure that you keep enough decimal paces in your calculation.

3. (0 pt) Consider two random variables X and Y whose $\mathbf{Q}=\begin{bmatrix}0.4 & 0.6\\ 0.4 & 0.6\end{bmatrix}$. Find $I(X;Y)$.

Note that the two rows in Q are identical. This means $Q(y|x)$ does not depend on x. In other words, knowing the value of X does not change the (conditional) pmf of Y. Therefore, X and Y are independent which implies $I(X;Y) = 0$.

See next page for a more direct solution.

Remark: Normally, to calculate $I(X;Y)$ you will need both $\underline{p}$ and Q. So, there must be something special about Q that allows you to get $I(X;Y)$ without $\underline{p}$.

Direct calculation:

$H(Y|x) = H([0.6 \ \ 0.4]) \approx 0.9710$ for any x

So, $H(Y|X) = \sum_x p(x) H(Y|x) \approx 0.9710 \underbrace{\sum_x p(x)}_{1} \approx 0.9710$

same!

$I(X;Y) = H(Y) - H(Y|X)$. So, we need $H(Y)$ which in turn need $\underline{q}(y)$

Let's try $p(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases}$

Then,

$$\overset{\underline{p}}{[1-p \quad p]} \overset{Q}{\begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}} = \overset{\underline{q}}{[0.6 \ \ 0.4]} \Rightarrow H(Y) = H([0.6 \ \ 0.4])$$

regardless of the value of p

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.

ECS 452: In-Class Exercise # **10**

Instructions

1. Separate into groups of no more than three persons. The group cannot be the same as any of your former groups. Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **08**/ **03** / 2019

Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. For each of the following DMC's probability transition matrices **Q**, (i) indicate whether the corresponding DMC is weakly symmetric (Yes or No), (ii) evaluate the corresponding capacity value (your answer should be of the form X.XXXX), and (iii) specify the channel input pmf (a row vector **p**) that achieves the capacity.

Check that
(1) all the rows of Q are permutations of each other
and
(2) all the column sums are equal

$-0.4\log_2 0.4 - 0.6\log_2 0.6$
≈ 0.9710

This is computed in the previous exercise already.

crossover probability p **Q**	Weakly Symmetric?	C	**p**
$\begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$ This is the Q matrix for a BSC.	Yes. BSC is symmetric and hence weakly symmetric.	For BSC, binary entropy function $C = 1 - h(p)$ $= 1 - h(0.4)$ $\approx 1 - 0.9710$ ≈ 0.0290 [bpcu]	C is achieved by uniform X on $\mathcal{X}$ $\underline{p} = [\frac{1}{2} \ \frac{1}{2}]$
$\underline{r}$ $\begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{bmatrix}$	① ✓ ② ✓ Yes	$\log_2 \lvert\mathcal{Y}\rvert - H(\underline{r})$ $= \log_2 4 - H([0.5 \ 0.5])$ $= 2 - 1 = 1$ [bpcu]	C is achieved by uniform X on $\mathcal{X}$ $\underline{p} = [\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}]$
$\begin{bmatrix} 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \end{bmatrix}$	① ✗ ② ✗ No	Note that there is only one non-zero element in each column ⇒ This is NO² channel $\Rightarrow C = \log_2 \lvert\mathcal{X}\rvert$ $= \log_2 4$ ≈ 2 [bpcu]	⇒ C is achieved by uniform X $\underline{p} = [\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}]$
$\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$	① ✓ ② ✗ No	Note that all the rows of Q are the same $\Rightarrow Q(y\|x)$ does not depend on x $\Rightarrow X \perp\!\!\!\perp Y$ $\Rightarrow I(X;Y) = 0$ for any $p(x)$ $\Rightarrow C = 0.0000$ [bpcu]	Any $\underline{p}$ will give the same $I(X;Y) = C = 0$.

Specifically, any $\underline{p} = [p_1 \ p_2]$ such that $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$ will work.