Name ________________________ ID3 _ _ _

ECS 452: Digital Communication Systems **2018/2**

HW 8 — Due: May 3, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 8 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider a convolutional encoder whose trellis diagram is given in Figure 8.1.

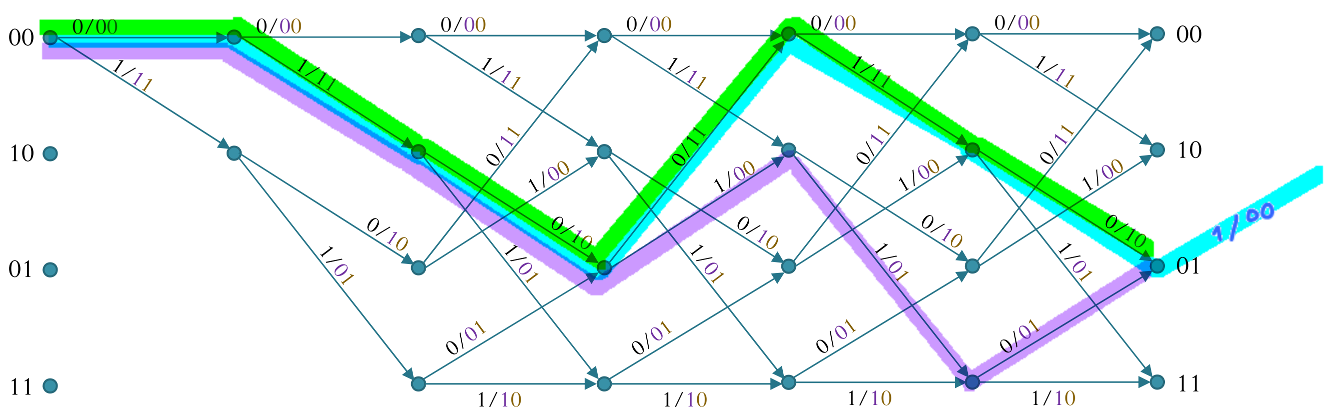


Figure 8.1: State diagram for a convolutional encoder

(a) Find the code rate $\frac{\text{one bit}}{\text{two bits}} = \frac{1}{2}$

(b) Suppose the data bits (message) are $\underline{\mathbf{b}} = [0100101]$. Find the corresponding codeword $\underline{\mathbf{x}}$.

From the blue highligthed path, we have $\underline{x}$ = [00111011111000]

(c) Find the data vector $\underline{\mathbf{b}}$ which gives the codeword $\underline{\mathbf{x}} = [001110111110]$.

From the green highlighted path, we have $\underline{b}$ = [010010]

Alternatively, because the codeword here is the same as the first 12 bits of the codeword in the previous part, we know that the data vector must be the same as the first 6 bits of the data vector from the previous part.

(d) Suppose that we observe $\underline{\mathbf{y}} = [001110000101]$ at the input of the minimum distance decoder. Explain why we can easily find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$ without applying the Viterbi algorithm.

From the purple highligthed path, we see that $\underline{y}$ itself is a codeword.
So, there is a codeword with distance = 0 from $\underline{y}$.
There can not be any codeword with smaller distance from $\underline{y}$ than 0.
So, $\hat{\underline{x}} = \underline{y}$.
From the purple highligthed path, we can also read $\hat{\underline{b}}$ = [010110]

(e) Suppose that we observe $\underline{\mathbf{y}} = [010101111110]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$. Show your work on Figure 7.2 below.

$\hat{\underline{x}}$ = [110101111110]

$\hat{\underline{b}}$ = [110010]

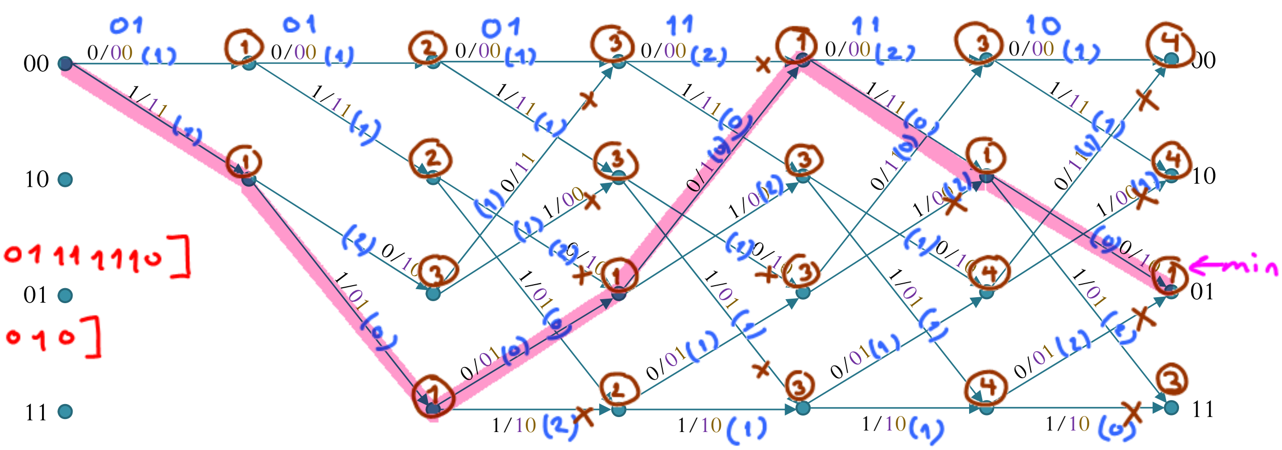


Figure 8.2: State diagram for a convolutional encoder

Make sure that all the running (cumulative) path metric are shown and the discarded branches are indicated at every steps.

Problem 2. Consider four vectors:

$$\mathbf{v}^{(1)} = \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} +1 \\ +1 \\ 0 \\ +1 \\ 0 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} +2 \\ 0 \\ +1 \\ +1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} +3 \\ +1 \\ +1 \\ +2 \\ -1 \end{pmatrix}.$$

Now, consider two vectors:

$$\mathbf{e}^{(1)} = \frac{1}{2}\begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \\ -1 \end{pmatrix}, \text{ and } \mathbf{e}^{(2)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

(a) Show that the two vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are orthonormal.

$\|\vec{e}^{(1)}\|^2 = \left(\frac{1}{2}\right)^2\left(1^2+(-1)^2+1^2+0^2+(-1)^2\right) = \frac{1}{4}(1+1+1+0+1) = 1 \Rightarrow \|\vec{e}^{(1)}\| = 1$

$\|\vec{e}^{(2)}\|^2 = \left(\frac{1}{\sqrt{3}}\right)^2\left(1^2+1^2+0^2+1^2+0^2\right) = \frac{1}{3}(3) = 1 \Rightarrow \|\vec{e}^{(2)}\| = 1$

Both of them have unit length. → orthonormal

$\langle \vec{e}^{(1)}, \vec{e}^{(2)} \rangle = \frac{1}{2} \times \frac{1}{\sqrt{3}} \times \left((1)(1) + (-1)(1) + (1)(0) + (0)(1) + (-1)(0)\right)$

$= \frac{1}{2\sqrt{3}}(1-1+0+0-0) = \frac{1}{2\sqrt{3}} \times 0 = 0$ $\Rightarrow$ They are orthogonal.

(b) Find the corresponding vectors $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, $\mathbf{c}^{(3)}$, and $\mathbf{c}^{(4)}$ that represent $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$, and $\mathbf{v}^{(4)}$ in the new coordinate system defined by vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$.

$\langle \vec{v}^{(1)}, \vec{e}^{(1)} \rangle = \frac{1}{2}(1+1+1+0+1) = \frac{4}{2} = 2$
$\langle \vec{v}^{(1)}, \vec{e}^{(2)} \rangle = \frac{1}{\sqrt{3}}(1-1+0+0+0) = 0$
$\Rightarrow \vec{c}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\langle \vec{v}^{(2)}, \vec{e}^{(1)} \rangle = \frac{1}{2}(1-1+0+0+0) = 0$
$\langle \vec{v}^{(2)}, \vec{e}^{(2)} \rangle = \frac{1}{\sqrt{3}}(1+1+0+1+0) = \frac{3}{\sqrt{3}} = \sqrt{3}$
$\Rightarrow \vec{c}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$

$\langle \vec{v}^{(3)}, \vec{e}^{(1)} \rangle = \frac{1}{2}(2+0+1+0+1) = \frac{4}{2} = 2$
$\langle \vec{v}^{(3)}, \vec{e}^{(2)} \rangle = \frac{1}{\sqrt{3}}(2+0+0+1+0) = \frac{3}{\sqrt{3}} = \sqrt{3}$
$\Rightarrow \vec{c}^{(3)} = \begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix}$

$\langle \vec{v}^{(4)}, \vec{e}^{(1)} \rangle = \frac{1}{2}(3-1+1+0+1) = \frac{4}{2} = 2$
$\langle \vec{v}^{(4)}, \vec{e}^{(2)} \rangle = \frac{1}{\sqrt{3}}(3+1+0+2+0) = \frac{6}{\sqrt{3}} = 2\sqrt{3}$
$\Rightarrow \vec{c}^{(4)} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$

Problem 3. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 8.4. Note that V and T_b are some positive constants. Your answers should be given in terms of them.

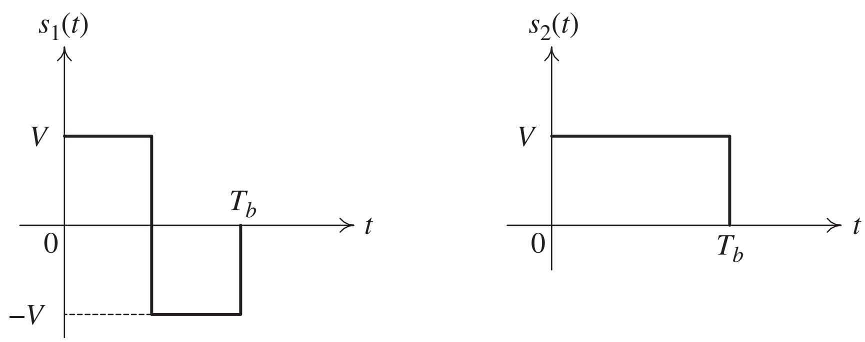


Figure 8.4: Signal set for Problem 3

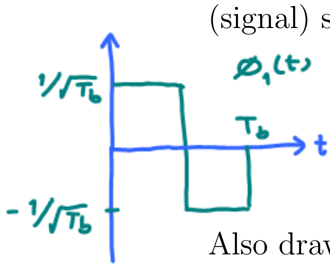
(a) Find the energy in each signal.

$$E_1 = E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

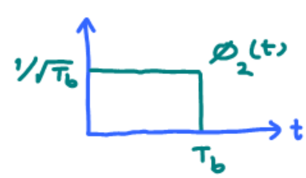
$$E_2 = E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$\equiv E_0$

(b) Find the two vectors that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space with respect to the following orthonormal vectors:



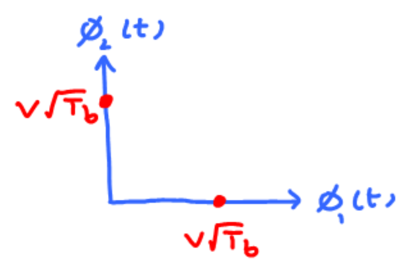
$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 \le t \le \frac{T_b}{2}, \\ -\frac{1}{\sqrt{T_b}}, & \frac{T_b}{2} < t < T_b, \\ 0, & \text{otherwise} \end{cases} \qquad \phi_2(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 \le t \le T_b, \\ 0, & \text{otherwise} \end{cases}$$



Also draw the corresponding constellation.

$$\left.\begin{aligned} \langle s_1(t), \phi_1(t) \rangle &= \int_{-\infty}^{\infty} s_1(t)\phi_1(t)\,dt = (V)\left(\tfrac{1}{\sqrt{T_b}}\right)\left(\tfrac{T_b}{2}\right) + (-V)\left(-\tfrac{1}{\sqrt{T_b}}\right)\left(\tfrac{T_b}{2}\right) = V\sqrt{T_b} \\ \langle s_1(t), \phi_2(t) \rangle &= \left(\tfrac{1}{\sqrt{T_b}}\right)\left(V\tfrac{T_b}{2} + (-V)\tfrac{T_b}{2}\right) = 0 \end{aligned}\right\} \vec{s}^{(1)} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \langle s_2(t), \phi_1(t) \rangle &= V\left(\left(\tfrac{1}{\sqrt{T_b}}\right)\tfrac{T_b}{2} + \left(-\tfrac{1}{\sqrt{T_b}}\right)\tfrac{T_b}{2}\right) = 0 \\ \langle s_2(t), \phi_2(t) \rangle &= V\left(\tfrac{1}{\sqrt{T_b}}\right)(T_b) = V\sqrt{T_b} \end{aligned}\right\} \vec{s}^{(2)} = \begin{pmatrix} 0 \\ V\sqrt{T_b} \end{pmatrix}$$



Problem 4. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 8.5. Note that V, α and T_b are some positive constants.

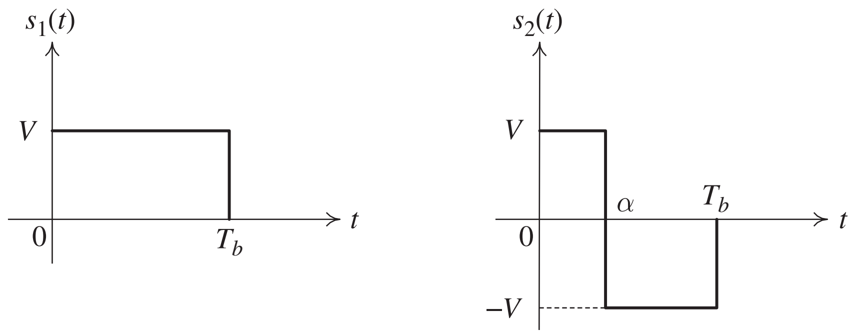


Figure 8.5: Signal set for Problem 4

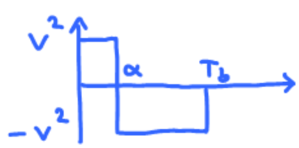
(a) Find the energy in each signal.

$$E_1 = E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$$E_2 = E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$\equiv E_s$

(b) Find $\langle s_1(t), s_2(t) \rangle$.

$= \int$ 

$dt = \alpha V^2 - (T_b - \alpha)V^2 = 2\alpha V^2 - T_b V^2$

(c) Consider two orthonormal vectors:

$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 < t < T_b, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi_2(t) = \frac{1}{\sqrt{\alpha\left(1 - \frac{\alpha}{T_b}\right)}} \times \begin{cases} 1 - \frac{\alpha}{T_b} & 0 < t \le \alpha, \\ -\frac{\alpha}{T_b}, & \alpha < t < T_b, \\ 0, & \text{otherwise} \end{cases}$$

Let's refer to this as β

(i) Check that they are orthonormal.

$$\|\phi_1\|^2 = \int_{-\infty}^{\infty} (\phi_1(t))^2 dt = \left(\frac{1}{\sqrt{T_b}}\right)^2 \times T_b = 1$$

$$\|\phi_2\|^2 = \int_{-\infty}^{\infty} (\phi_2(t))^2 dt = \frac{1}{\cancel{\alpha}\cancel{(1-\frac{\alpha}{T_b})}} \left((1-\frac{\alpha}{T_b})^{\cancel{2}} \cancel{\alpha} + (+\frac{\alpha}{T_b})^{\cancel{2}} (\cancel{T_b}-\alpha) \right) = 1$$

$$\langle \phi_1(t), \phi_2(t) \rangle = \int_{-\infty}^{\infty} \phi_1(t)\phi_2(t)dt = \frac{1}{\sqrt{T_b}} \beta \left((1-\frac{\alpha}{T_b})\alpha + (-\frac{\alpha}{T_b})(T_b-\alpha) \right) = 0$$

(ii) Find the two vectors $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space with respect to the given orthonormal vectors.

$$\langle s_1(t), \phi_1(t) \rangle = \int_{-\infty}^{\infty} s_1(t)\phi_1(t)dt = \int_0^{T_b} \frac{V}{\sqrt{T_b}} dt = \frac{V}{\sqrt{T_b}} T_b = V\sqrt{T_b}$$

$$\langle s_1(t), \phi_2(t) \rangle = \int_{-\infty}^{\infty} s_1(t)\phi_2(t)dt = \beta V \left((1-\frac{\alpha}{T_b})\alpha + (-\frac{\alpha}{T_b})(T_b-\alpha) \right) = 0$$

$$\vec{s}^{(1)} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}.$$

$$\langle s_2(t), \phi_1(t) \rangle = \int_{-\infty}^{\infty} s_2(t)\phi_1(t)dt = \frac{1}{\sqrt{T_b}} \left(V\alpha + (-V)(T_b-\alpha) \right) = \frac{V}{\sqrt{T_b}}(2\alpha - T_b)$$

$$= V\sqrt{T_b}\left(2\frac{\alpha}{T_b} - 1\right)$$

$$\langle s_2(t), \phi_2(t) \rangle = \int_{-\infty}^{\infty} s_2(t)\phi_2(t)dt = \beta \left(V(1-\frac{\alpha}{T_b})(\alpha) + (+V)(+\frac{\alpha}{T_b})(T_b-\alpha) \right)$$

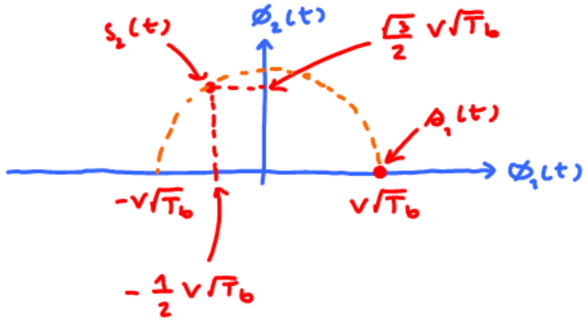
$$= 2\beta V (1-\frac{\alpha}{T_b})\alpha = V\sqrt{T_b}\, 2\sqrt{\frac{\alpha}{T_b}(1-\frac{\alpha}{T_b})}$$

$$\vec{s}^{(2)} = V\sqrt{T_b} \times \begin{pmatrix} 2\frac{\alpha}{T_b} - 1 \\ 2\sqrt{\frac{\alpha}{T_b}(1-\frac{\alpha}{T_b})} \end{pmatrix} = \sqrt{E_s} \begin{pmatrix} 2r-1 \\ 2\sqrt{r(1-r)} \end{pmatrix}$$

where $r = \frac{\alpha}{T_b}$.

(iii) Draw the corresponding constellation when $\alpha = \frac{T_b}{4}$.

When $\alpha = \frac{T_b}{4}$, $s_2(t) = 2V\sqrt{\frac{T_b}{4}(\frac{3}{4})}\, \phi_2(t) - (1-\frac{1}{2})V\sqrt{T_b}\, \phi_1(t) = \frac{\sqrt{3}}{2}V\sqrt{T_b}\, \phi_2(t) - \frac{1}{2}V\sqrt{T_b}\, \phi_1(t)$

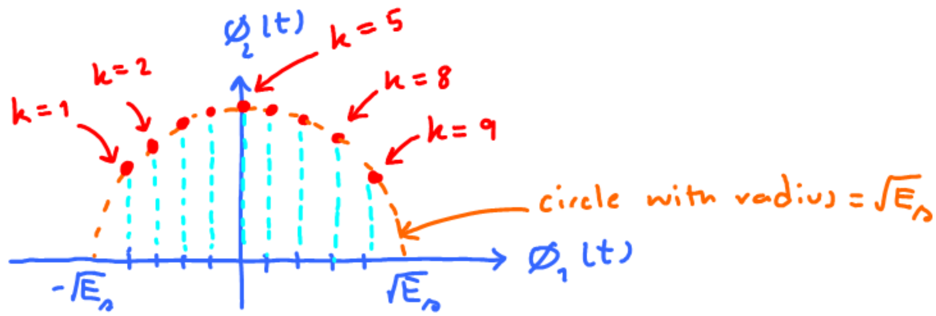


(iv) Draw $\mathbf{s}^{(2)}$ when $\alpha = \frac{k}{10}T_b$ where $k = 1, 2, \ldots, 9$.

Note that

$$(2r-1)^2 + \left(2\sqrt{r(1-r)}\right)^2 = 4r^2 - 4r + 1 + 4r - 4r^2 = 1.$$

$$r = \frac{\alpha}{T_b} = \frac{k}{10}$$



Extra Question

Here is an optional question for those who want more practice.

Problem 5. Consider a convolutional code generated by the encoder shown in Figure 8.3. Suppose that we observe $\underline{\mathbf{y}} = [110111000110]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\underline{\hat{\mathbf{x}}}$ and the decoded message $\underline{\hat{\mathbf{b}}}$. Caution: The trellis diagram is not the same as the one used in Problem 1.

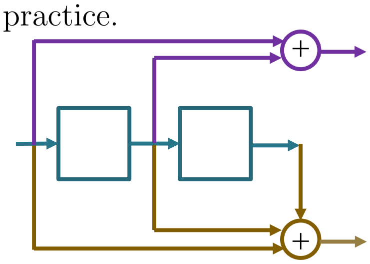


Figure 8.3: Encoder for Problem 5

Solution: Observe that the circuit diagram is exactly the same as the one used in Exercise 15. We have already found its state diagram; this is shown in Figure 8.6a. From the state diagram, we can then create the code trellis shown in Figure 8.6b.

For Viterbi decoding, the trellis diagram is shown in Figure 8.7. (Don't forget that the trellis diagram always starts with the all-zero state.)

Tracing back the trellis diagram, we get $\boxed{\underline{\hat{\mathbf{b}}} = [101110]}$ and $\boxed{\underline{\hat{\mathbf{x}}} = [111110000110]}$.

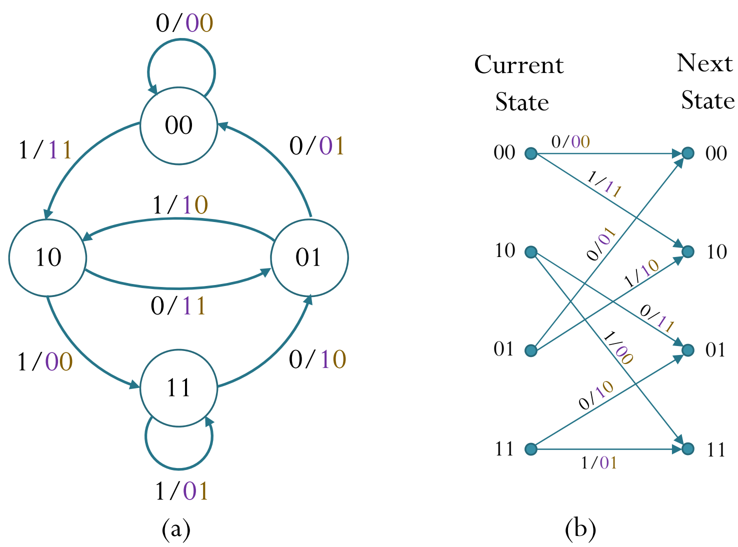


Figure 8.6: (a) State diagram and (b) Code Trellis for Problem 5

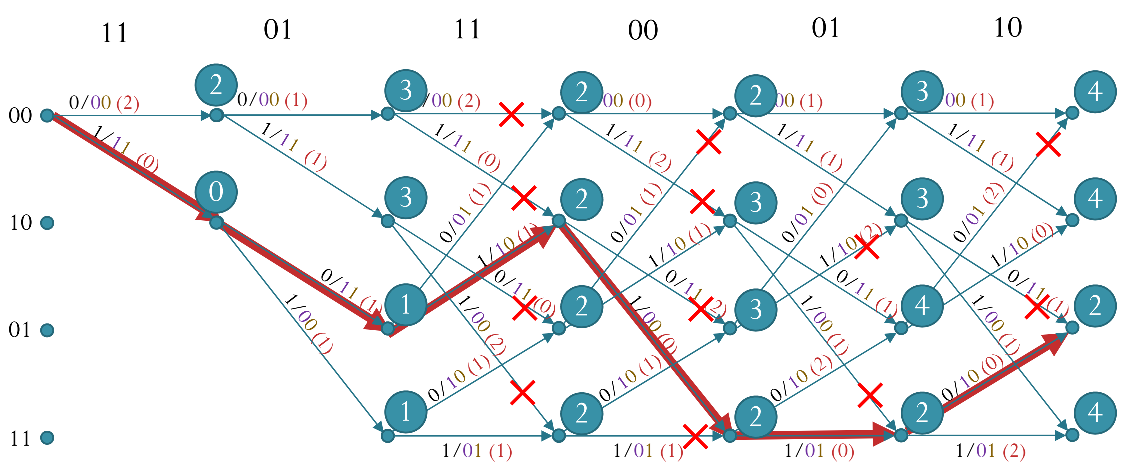


Figure 8.7: Trellis diagram for Problem 5.