ECS 452: Digital Communication Systems **2018/2**

HW 7 — Due: April 26, 4 PM

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Instructions

(a) This assignment has 5 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Continue from the previous assignment. Consider a block code whose generator matrix is

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(a) Suppose we receive $\underline{\mathbf{y}} = [1\,1\,1\,1\,0\,1]$.

(i) Minimum distance decoding:

i. Find the distance $d(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ between this received vector $\underline{\mathbf{y}}$ and each of the possible codewords $\underline{\mathbf{x}}$.

$\underline{y}$ = 1 1 1 1 0 1

$\underline{\mathbf{b}}$			$\underline{\mathbf{x}}$						$d(\underline{\mathbf{x}}, \underline{\mathbf{y}})$
0	0	0	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	0	$\underline{0}$	5
0	0	1	$\underline{0}$	$\underline{0}$	1	1	$\underline{1}$	$\underline{0}$	4
0	1	0	$\underline{0}$	1	$\underline{0}$	$\underline{0}$	$\underline{1}$	1	4
0	1	1	$\underline{0}$	1	1	1	0	1	1
1	0	0	1	$\underline{0}$	$\underline{0}$	1	0	1	2
1	0	1	1	$\underline{0}$	1	$\underline{0}$	$\underline{1}$	1	3
1	1	0	1	1	$\underline{0}$	1	$\underline{1}$	$\underline{0}$	3
1	1	1	1	1	1	$\underline{0}$	0	$\underline{0}$	2

We underline the bits in $\underline{x}$ that are different from the corresponding bits in $\underline{y}$.

ii. Use the answer in the previous part to find $\hat{\underline{\mathbf{x}}}$ and $\hat{\underline{\mathbf{b}}}$

$$\hat{\underline{x}} = \arg\min_{\underline{x}} d(\underline{x}, \underline{y}) = [\underbrace{0\ 1\ 1}\ 1\ 0\ 1] \Rightarrow \hat{\underline{b}} = [0\ 1\ 1]$$

The generator matrix has the identity matrix of size 3x3 in the front. So the message is the first three bits in the front.

(ii) Decoding via the syndrome:

i. Find the parity check matrix $\mathbf{H}$ of this code.

$$G = \left[\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}\ \underbrace{\begin{matrix}1&0&1\\0&1&1\\1&1&0\end{matrix}}_{P}\right] \Rightarrow H = \left[\underbrace{\begin{matrix}1&0&1\\0&1&1\\1&1&0\end{matrix}}_{P^T}\ \underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}\right]$$

ii. Find the syndrome vector $\underline{\mathbf{s}}$.

$\underline{y} = 1\ 1\ 1\ 1\ 0\ 1$

$$\begin{bmatrix}1&0&1&1&0&0\\0&1&1&0&1&0\\1&1&0&0&0&1\end{bmatrix}\begin{matrix}\xrightarrow{\Sigma} 1\\ \xrightarrow{\Sigma} 0\\ \xrightarrow{\Sigma} 1\end{matrix}$$

$\underline{s} = \underline{y} H^T$

$= (\text{sum of all columns of } H \text{ except the } 5^{th} \text{ column})^T$

$= [1\ 0\ 1]$

iii. Use the answer in the previous parts to find $\hat{\underline{\mathbf{x}}}$ and $\hat{\underline{\mathbf{b}}}$

$\underline{s}$ is the same as the first column of H. Hence, $\hat{\underline{e}} = [1\ 0\ 0\ 0\ 0\ 0]$.

$$\hat{\underline{x}} = \underline{y} \oplus \hat{\underline{e}} = [\underbrace{0\ 1\ 1}\ 1\ 0\ 1] \Rightarrow \hat{\underline{b}} = [0\ 1\ 1]$$

Problem 2. Consider a (15,11) Hamming code.

(a) Find the length of each codeword.

$n = 15$

(b) For each codeword, how many bits are message bits?

$k = 11$

Remark:

code rate $= \frac{k}{n} = \frac{11}{15}$

(c) Suppose ~~the code~~ each codeword is constructed so that the parity bits are in the front and the message bits are in the back. Given an example of a generator matrix $\mathbf{G}$ and a corresponding parity check matrix $\mathbf{H}$ for such (15,11) Hamming code.

(15,11) Hamming code
$\left.\begin{array}{l} k=11 \\ n=15 \end{array}\right\} \Rightarrow m=4.$

First, we construct the parity check matrix H.

$m=4 \Rightarrow$ Put all $2^4-1=15$ nonzero 4-bit column vectors as columns of H. They can be arranged in any order but for simplicity, we put the columns corresponding to the identity matrix in the front:

$$\mathbf{H} = \left(\begin{array}{cccc|ccccccccccc}
1&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&1&0&0&1&1&1&0&0&0&0&1&1&1&1\\
0&0&1&0&0&1&1&1&0&1&1&0&0&1&1\\
0&0&0&1&1&0&1&1&1&0&1&0&1&0&1
\end{array}\right)$$

$\underbrace{}_{I_4} \; \underbrace{}_{P}$

From the recipe discussed in class,

when $H = [I \,;A]$,

we know that the generator matrix is

$G = [A^T ; I]$

the size of this I is easily determined from the # rows in A^T

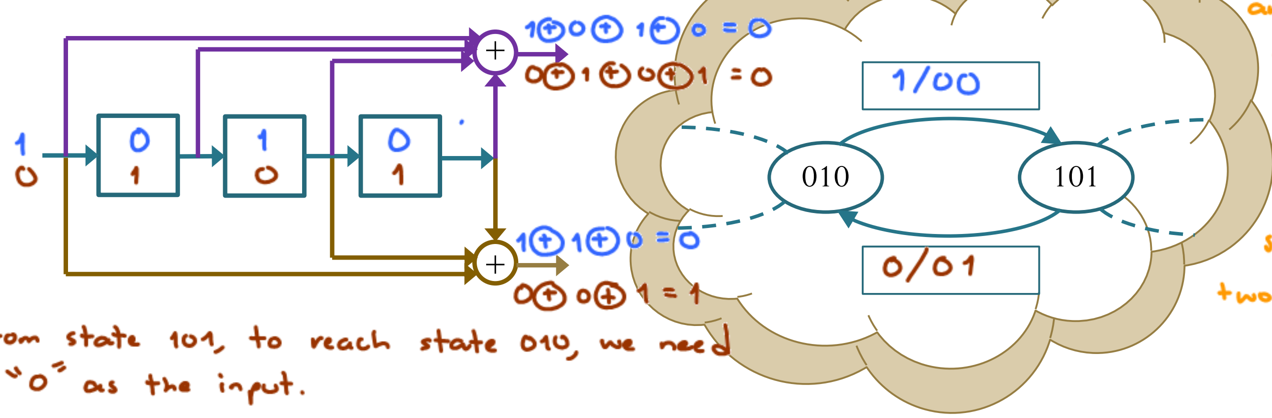
P^T I_{11}

$$\mathbf{G} = \left(\begin{array}{cccc|ccccccccccc}
0&1&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&1&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&1&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&1&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&1&0&0&0&0&0\\
1&0&1&1&0&0&0&0&0&0&1&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&1&0&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&1&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&1&0\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)$$

Problem 3. Consider a convolutional encoder whose circuit diagram and a part of the corresponding state diagram is given in Figure 7.1. Write the suitable labels for the two arrows shown in the state diagram.

From state 010, to reach state 101, we need a "1" as the input.

There are 3 FFs. Therefore, there are $2^3 = 8$ possible states. However, here, we show only two of them.



From state 101, to reach state 010, we need a "0" as the input.

Figure 7.1: Circuit diagram and a part of the corresponding state diagram for a convolutional encoder. Only two states are shown in the state diagram.

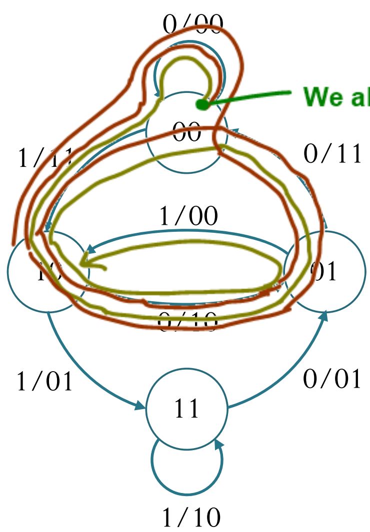


Figure 7.2: State diagram for a convolutional encoder

We always start from state $\underline{0}$.

Problem 4. Consider a convolutional encoder whose state diagram is given in Figure 7.2.

(a) Find the code rate

The label on each arrow is of the form X/XX

one bit $\Rightarrow k=1$

two bits $\Rightarrow n=2$

code rate $= \frac{k}{n} = \frac{1}{2}$

(b) Suppose the data bits (message) are 0100101. Find the corresponding codeword.

From the trace on the state diagram, we have $\underline{x}$ = [00111011111000]

(c) Find the data vector $\underline{\mathbf{b}}$ which gives the codeword $\underline{\mathbf{x}} = [001110111110110011]$

From the trace on the state diagram, we have $\underline{b}$ = [010010001]

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. In the previous assignment, we consider the following encoding for a systematic linear block code:

- The bit positions that are powers of 2 (1, 2, 4, 8, 16, etc.) are check bits.
- The rest (3, 5, 6, 7, 9, etc.) are filled up with the k data bits.
- Each check bit forces the parity of some collection of bits, including itself, to be even.
  - To see which check bits the data bit in position i contributes to, rewrite i as a sum of powers of 2. A bit is checked by just those check bits occurring in its expansion.

For the case when the codeword's length $n = 7$, we found that

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(a) Explain, from the elements inside the matrix **H**, how this is a Hamming code.

The columns of H cover all the nonzero binary vectors of length 3.

(b) Consider the following decoding instruction:

- When a vector is observed, the receiver initializes a counter to zero. It then examines each check bit at position i (i = 1, 2, 4, 8, ...) to see if it gives the correct parity.
- If not, the receiver adds i to the counter. If the counter is zero after all the check bits have been examined (i.e., if they were all correct), the observed vector is accepted as a valid codeword. If the counter is nonzero, it contains the position of the incorrect bit.

Explain how the instruction above is the "same" as the decoding via the syndrome described in class.

The provided decoding instruction starts with calculations of the bits inside the syndrome $\underline{s} = \underline{y}\mathbf{H}^T$.
Note that p_j is at position $i = 2^{j-1}$.

If s_j is 1 (received vector does not satisfy eq. (j)), the value $i = 2^{j-1}$ is added to the counter.
At the end, the value in the counter is $I = \sum_j s_j 2^{j-1}$ which is exactly the same as converting the binary vector $\underline{s}$ to a decimal number.

Note that the columns of H are arranged in increasing values (treating the top element as the LSB).
Therefore, the value I in the counter simply indicates the index of the column of H that matches $\underline{s}$.
We know that this is the most-likely position of the incorrect bit.

Problem 6. Consider a block code whose generator matrix is

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

A valid codeword was transmitted and potentially corrupted by the channel. Suppose that, at the decoder, the syndrome is found to be $\underline{\mathbf{s}} = [110]$. Find <u>all</u> the error patterns $\underline{\mathbf{e}}$ that can cause this syndrome.

At the decoder, the syndrome is calculated by

$$\underline{s} = \underline{y} H^T$$

In class, we've shown that the syndrome only responses to the error pattern:

$$\underline{s} = \underbrace{(\underline{x} \oplus \underline{e})}_{\underline{y}} H^T = \underline{b}\underbrace{GH^T}_{\underline{0}} \oplus \underline{e} H^T = \underline{e} H^T.$$

Because G is given in the form $[P \,\vdots\, I]$, we can easily find H by

$$H = [I \,\vdots\, P^T] = \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \; \overbrace{\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}}^{P^T}\right].$$

We already know that $\underline{s} = [1\ 1\ 0]$ and we want to solve for $\underline{e}$.

From $\underline{s} = \underline{e} H^T$,

substituting the values of $\underline{s}$ and H gives

$$[1\ 1\ 0] = [e_1\ e_2\ e_3\ e_4\ e_5\ e_6] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

So, we have three equations:

$$e_1 \oplus e_4 \oplus e_6 = 1$$
$$e_2 \oplus e_4 \oplus e_5 = 1$$
$$e_3 \oplus e_5 \oplus e_6 = 0$$

Note that because we have six variables $(e_1, e_2, \ldots, e_6)$ but only three equations, there are three free variables. Here, we will choose e_4, e_5, and e_6 to be free.

From $\left\{\begin{aligned} e_1 &= 1 \oplus e_4 \oplus e_6 \\ e_2 &= 1 \oplus e_4 \oplus e_5 \\ e_3 &= 0 \oplus e_5 \oplus e_6 \end{aligned}\right\}$, we can list all the possible solutions by plugging in all possible values of e_4, e_5, and e_6:

e_4	e_5	e_6	e_1	e_2	e_3
0	0	0	1	1	0
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	1	0	1
1	1	0	0	1	1
1	1	1	1	1	0

$\Rightarrow$

$e_1\, e_2\, e_3\, e_4\, e_5\, e_6$

[1 1 0 0 0 0]

[0 1 1 0 0 1]

[1 0 1 0 1 0]

[0 0 0 0 1 1]

[0 0 0 1 0 0]

[1 0 1 1 0 1]

[0 1 1 1 1 0]

[1 1 0 1 1 1]

Remark: Using the recipe from our lectures, we see that the syndrome is the same as the 4^{th} column of H. Therefore, the (min-distance) decoder should conclude that $\hat{\underline{e}} = [0\ 0\ 0\ 1\ 0\ 0]$.
Here, we found all the possible error patterns. Observe that $[0\ 0\ 0\ 1\ 0\ 0]$ is the error pattern with min weight. Therefore, it is the error pattern that gives the codeword with min distance from the received vector at the decoder. So, our in-class recipe does choose the optimal error pattern.