Name ______________________ ID3 _ _ _

ECS 452: Digital Communication Systems **2018/2**

HW 3 — Due: Feb 22, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 6 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

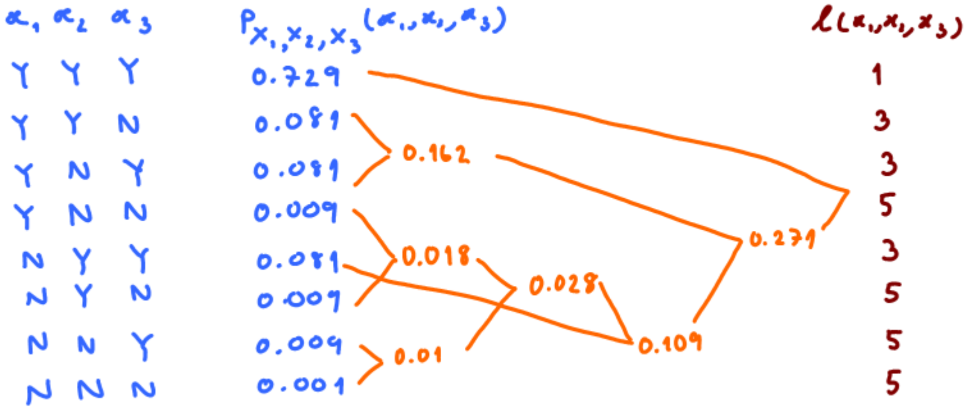
(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Continue from the Example 2.40 in the lecture. A memoryless source emits two possible message Y(es) and N(o) with probability 0.9 and 0.1, respectively.

(a) Calculate the entropy (per source symbol) of this source.

$$H(X) = -\sum_{x} p_X(x)\log_2 p_X(x) \underset{\substack{\uparrow \\ \text{MATLAB}}}{=} 0.469 \text{ bits per symbol}$$

(b) Find the expected codeword length per symbol of the Huffman binary code for the third-order extensions of this source.



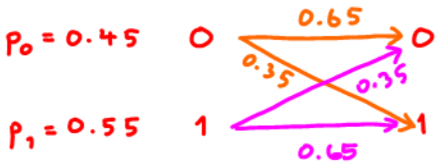
x_1	x_2	x_3	$p_{X_1,X_2,X_3}(x_1,x_2,x_3)$	$\ell(x_1,x_2,x_3)$
Y	Y	Y	0.729	1
Y	Y	N	0.081	3
Y	N	Y	0.081	3
Y	N	N	0.009	5
N	Y	Y	0.081	3
N	Y	N	0.009	5
N	N	Y	0.009	5
N	N	N	0.001	5

$\mathbb{E}[\ell(X_1,X_2,X_3)] = 1.5980$ bits per three source symbols $\Rightarrow L_3 = 0.5327$ bits per source symbol

Problem 2. Consider a BSC whose crossover probability for each bit is $p = 0.35$. Suppose $P[X = 0] = 0.45$. $\equiv p_0$ $\quad 1-p = 0.65$

$p_1 = 1 - p_0 = 1 - 0.45 = 0.55$

(a) Draw the channel diagram.



(b) Find the channel matrix **Q**.

Method 2: Directly read the transition (conditional) probabilities from the channel diagram.

Method 1

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 1-p & p \\ 1 & p & 1-p \end{array} = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.35 & 0.65 \end{array}$$

(c) Find the joint pmf matrix **P**.

the row vector $\underline{p}$ of input probabilities: $\underline{p} = \left[P[X=0] \quad P[X=1]\right] = [p_0 \quad p_1] = [0.45 \quad 0.55]$

To get the matrix P, we simply scale each row of matrix Q by the corresponding p(x)

$$Q = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.35 & 0.65 \end{array} \begin{array}{c} \xrightarrow{\times 0.45} \\ \xrightarrow[\times 0.55]{} \end{array} \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.2925 & 0.1575 \\ 1 & 0.1925 & 0.3575 \end{array} = P$$

(d) Find the row vector $\underline{q}$ which contains the pmf of the channel output Y.

$\downarrow \Sigma \quad \downarrow \Sigma$

Method 1: $\underline{q} = [0.485 \quad 0.515]$

Method 2: $\underline{q} = \underline{p}Q = [0.45 \quad 0.55]\begin{bmatrix} 0.65 & 0.35 \\ 0.35 & 0.65 \end{bmatrix} = [0.485 \quad 0.515]$

(e) Analyze the performance of all four reasonable detectors for this binary channel. Complete the table below:

$\hat{x}(y)$	$\hat{x}(0)$	$\hat{x}(1)$	$P(\mathcal{C})$	$P(\mathcal{E})$ $= 1 - P(\mathcal{C})$
y	0	1	$0.65(p_0+p_1) = 0.65$	0.35
$1-y$	1	0	$0.35(p_0+p_1) = 0.35$	0.65
1	1	1	$(0.35+0.65)p_1 = p_1 = 0.55$	0.45
0	0	0	$(0.65+0.35)p_0 = p_0 = 0.45$	0.55

The value of $\hat{x}(y)$ tell us which row to select in the column y of the P matrix.

$$Q = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

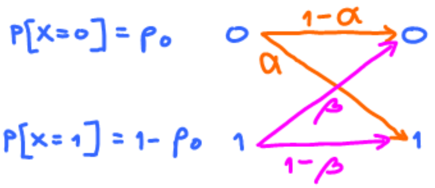
$$= \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.35 & 0.65 \end{array} \begin{array}{c} \xrightarrow{\times p_0} \\ \xrightarrow[\times p_1]{} \end{array} \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & \boxed{0.65p_0} & \boxed{0.35p_0} \\ 1 & \boxed{0.35p_1} & \boxed{0.65p_1} \end{array} = P$$

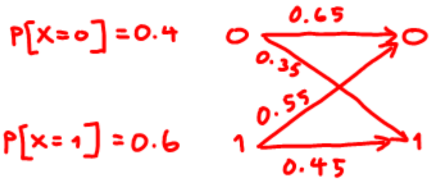
$Q(0|0) = 0.65, \quad Q(1|1) = 0.45$

Problem 3. Consider a BAC whose $Q(1|0) = 0.35$ and $Q(0|1) = 0.55$. Suppose $P[X = 0] = 0.4$. $= p_0 \Rightarrow p_1 = 1 - 0.4 = 0.6$

(a) Draw the channel diagram.

In some of these parts, we will also try to derive the answer in a general form.
So, we will start with $Q(1|0) = \alpha$ and $Q(0|1) = \beta$.





(b) Find the joint pmf matrix $\mathbf{P}$.

To get the matrix P, we simply scale each row of matrix Q by the corresponding p(x).

First, we find the Q matrix,

$$Q = \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \beta & 1-\beta \end{array} \quad \begin{array}{c} \xrightarrow{\times p_0} \\ \xrightarrow[\times 1-p_0]{} \end{array} \quad \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & p_0(1-\alpha) & p_0\alpha \\ 1 & (1-p_0)\beta & (1-\beta)(1-p_0) \end{array} = P$$

For this question,

$$Q = \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.55 & 0.45 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.4} \\ \xrightarrow[\times 0.6]{} \end{array} \quad \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 0.26 & 0.14 \\ 1 & 0.33 & 0.27 \end{array} = P$$

(c) Find the row vector $\underline{\mathbf{q}}$ which contains the pmf of the channel output Y.

Method 2: $\downarrow\Sigma \quad \downarrow\Sigma$

$$\begin{bmatrix} 0.59 & 0.41 \end{bmatrix}$$

Method 1: Recall that $\underline{q} = \underline{p}Q$.

For this question, $\underline{p} = \begin{bmatrix} p_0 & 1-p_0 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$.

Therefore, $\underline{q} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.26 & 0.14 \\ 0.33 & 0.27 \end{bmatrix} = \begin{bmatrix} 0.59 & 0.41 \end{bmatrix}$

(d) Analyze the performance of all four reasonable detectors for this binary channel. Complete the table below:

$\hat{x}(y)$	$\hat{x}(0)$	$\hat{x}(1)$	$P(\mathcal{C})$	$P(\mathcal{E})$ = $1 - P(\mathcal{C})$
y	0	1	0.26+0.27 = 0.53	0.47
$1-y$	1	0	0.33+0.14 = 0.47	0.53
1	1	1	0.33+0.27 = 0.60	0.40
0	0	0	0.26+0.14 = 0.40	0.60

The value of $\hat{x}(y)$ tells us which row to select in the column y of the P matrix.

$$Q = \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 0.65 & 0.35 \\ 1 & 0.55 & 0.45 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.4} \\ \xrightarrow[\times 0.6]{} \end{array} \quad \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 0.26 & 0.14 \\ 1 & 0.33 & 0.27 \end{array}$$

The convention for our class is that these numbers are ordered in the same way that they are specified in the support.

Problem 4. Consider a DMC whose $\mathcal{X} = \{1,2,3\}$, $\mathcal{Y} = \{1,2,3\}$, and

$$\mathbf{Q} = \begin{array}{c|ccc} x\backslash y & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.2 & 0.3 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.2 & 0.6 \end{array}.$$

Suppose the input probability vector is $\underline{\mathbf{p}} = [\overset{1}{0.2}, \overset{2}{0.4}, \overset{3}{0.4}]$.

(a) Find the joint pmf matrix $\mathbf{P}$.

We can get the P matrix by scaling each row of the Q matrix using the corresponding input probability p(x).

$$Q = \begin{array}{c|ccc} x\backslash y & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.2 & 0.3 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.2 & 0.6 \end{array} \begin{array}{c} \\ \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \begin{array}{c|ccc} x\backslash y & 1 & 2 & 3 \\ \hline 1 & 0.10 & 0.04 & 0.06 \\ 2 & 0.12 & 0.16 & 0.12 \\ 3 & 0.08 & 0.08 & 0.24 \end{array} = P$$

$$\Sigma\downarrow \qquad \Sigma\downarrow \qquad \Sigma\downarrow$$

(b) Find the row vector $\underline{\mathbf{q}}$ which contains the pmf of the channel output Y.

Method 1: $[0.3 \quad 0.28 \quad 0.42]$

Method 2: $\underline{q} = \underline{p}P = [0.2 \quad 0.4 \quad 0.4] \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = [\overset{1}{0.3} \quad \overset{2}{0.28} \quad \overset{3}{0.42}]$

(c) Find the following probabilities:

(i) $P[X = 1] = 0.2$

(ii) $P[Y = 2] = 0.28$

(iii) $P[X = 1, Y = 2] = 0.04$

(iv) $P[Y = 2|X = 1] = 0.2$

(v) $P[X = 1|Y = 2] = \overset{P(A|B)}{}\ \frac{P(A\cap B)}{P(B)} = \frac{P[X=1, Y=2]}{P[Y=2]} = \frac{0.04}{0.28} = \frac{4}{28} = \frac{1}{7}$

(vi) Find the error probability of the naive decoder. Naive Decoder : $\hat{x}_{Naive}(y) = y$

$$Q = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.2 & 0.3 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.2 & 0.6 \end{array} \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & \boxed{0.10} & 0.04 & 0.06 \\ 2 & 0.12 & \boxed{0.16} & 0.12 \\ 3 & 0.08 & 0.08 & \boxed{0.24} \end{array} = P$$

For naive decoder, look at each column of P and select (circle) the element whose corresponding x value is the same as y in that column.

$P(\mathcal{C}) = 0.10 + 0.16 + 0.24 = 0.5$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.5 = 0.5$

(vii) Find the error probability of the (DIY) decoder $\hat{x}(y) = 4 - y$. ⇒ Decoder table :

y	$\hat{x}(y)$
1	3
2	2
3	1

$$Q = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.2 & 0.3 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.2 & 0.6 \end{array} \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.4} \\ \xrightarrow{\times 0.4} \end{array} \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 1 & 0.10 & 0.04 & \boxed{0.06} \\ 2 & 0.12 & \boxed{0.16} & 0.12 \\ 3 & \boxed{0.08} & 0.08 & 0.24 \end{array} = P$$

For DIY decoder, look at each column of P and select the element whose corresponding x value is the same as $\hat{x}(y)$ in the decoder table.

$P(\mathcal{C}) = 0.08 + 0.16 + 0.06 = 0.30$

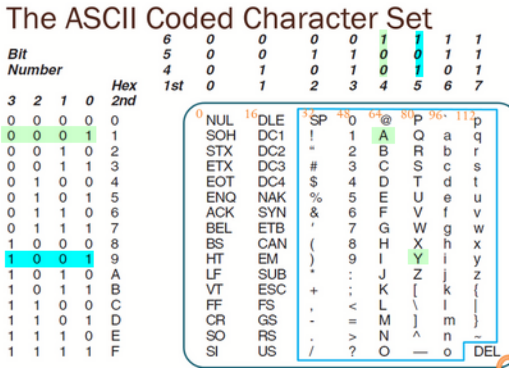
$P(\mathcal{E}) = 1 - 0.30 = 0.70$

Problem 5. A DMC has $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} = \{1, 2, 3\}$. The following decoding table is used to decode the channel output.

y	$\hat{x}(y)$
1	0
2	1
3	0

Change all '2' to '1'.
Change all '1' and '3' to '0'

Suppose the channel output string is 21221322113312212213 2.



The ASCII Coded Character Set

Bit Number					6	0	0	0	0	1	1	1	1
					5	0	0	1	1	0	0	1	1
					4	0	1	0	1	0	1	0	1
				Hex	1st	0	1	2	3	4	5	6	7
3	2	1	0	2nd		0	16	32	48	64	80	96	112
0	0	0	0	0		NUL	DLE	SP	0	@	P		p
0	0	0	1	1		SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2		STX	DC2	"	2	B	R	b	r
0	0	1	1	3		ETX	DC3	#	3	C	S	c	s
0	1	0	0	4		EOT	DC4	$	4	D	T	d	t
0	1	0	1	5		ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6		ACK	SYN	&	6	F	V	f	v
0	1	1	1	7		BEL	ETB	'	7	G	W	g	w
1	0	0	0	8		BS	CAN	(	8	H	X	h	x
1	0	0	1	9		HT	EM	)	9	I	Y	i	y
1	0	1	0	A		LF	SUB	*	:	J	Z	j	z
1	0	1	1	B		VT	ESC	+	;	K	[	k	{
1	1	0	0	C		FF	FS	,	<	L	\	l	|
1	1	0	1	D		CR	GS	-	=	M	]	m	}
1	1	1	0	E		SO	RS	.	>	N	^	n	~
1	1	1	1	F		SI	US	/	?	O	—	o	DEL

(a) Find the corresponding decoded string.

101100110000011011001

(b) Suppose the channel input string is produced from an ASCII source encoder by the command `dec2bin(SourceString,7)` in MATLAB. Assume that there is no channel decoding error. Find the corresponding source string.
(There is no need to actually use MATLAB to solve this problem.
Hint: See pages 19 and 28-29 of the slides for Chapter 2.)

YAY

Y A Y

Extra Questions

Here are some optional questions for those who want more practice.

Problem 6. ***Optimal code lengths that require one bit above entropy***: The source coding theorem says that the Huffman code for a random variable X has an expected length strictly less than $H(X) + 1$. Give an example of a random variable for which the expected length of the Huffman code (without any source extension) is very close to $H(X) + 1$.

We want to come up with some simple example. Therefore, we shall start by considering the Bernoulli RV X which has only two possible values:

Consider $X \sim \text{Bernoulli}(p_1)$: $p(x) = \begin{cases} p_1, & x = 1, \\ 1-p_1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$

Because there are only two possible values, the pairing in the Huffman coding process must be between these two values:

x	$p(x)$	$c(x)$	$\ell(x)$
0	$1-p_1$	0	1
1	p_1	1	1

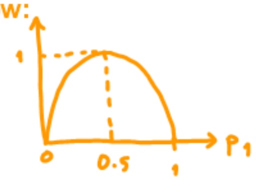
Therefore, Huffman coding (without extension) always needs 1 bit per symbol.

Now, we want the $\underbrace{\text{expected length}}_{=1}$ to be $\approx H(X)+1$

So, we must have $H(X) \approx 0$.

By definition, $H(X) = -\sum_x p(x) \log_2 p(x) = -(1-p_1)\log(1-p_1) - p_1 \log p_1$

You may recall that in Section 2.4, this expression is called the binary entropy function. The plot of this function is shown in the lecture notes; it is sketches below:



Note that to get this function to be ≈ 0, we need to consider $p_1 \approx 0$ or $p_1 \approx 1$.

Also note that we don't want to have $p_1 = 0$ or $p_1 = 1$ because they would make our RV X degenerated (deterministic). For degenerated RV, we don't have to waste any bit to convey its value. (The value is already pre-determined.) For example, if $p_1 = 1$, we know that $P[X = 1] = 1$ and hence $X \equiv 1$ all the time. There is no uncertainty. Anyone can guess the value of X with 100% accuracy by simply guessing the value 1 every time. Alternatively, we may think of the situation here as sending the empty string (ε) to the receiver.

So, $\mathbb{E}[\ell(X)] = \mathbb{E}[0] = 0$.

Because $H(X)$ is also 0, we have $\mathbb{E}[\ell(X)] = H(X)$ and not $H(X)+1$.

More formally we can take $\lim_{p_1 \to 0}$ or $\lim_{p_1 \to 1}$ on the function and show that $H(X) \to 0$.

$$\left(\lim_{x \to 0^+} x \ln x = \underset{\substack{\uparrow \\ \text{L'Hôpital's rule}}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} (-x) = 0 \right)$$

So, if p_1 is close to 1 or 0, the entropy will be almost 0.

But we still need $\mathbb{E}[\ell(X)] = 1$ bit to send X.

So, the expected length will be very close to $H(X)+1$.

Problem 7. Continue from the Example 2.40 in the lecture and Problem 1. A memoryless source emits two possible message Y(es) and N(o) with probability 0.9 and 0.1, respectively.

(a) Use `MATLAB` to find the expected codeword length **per (source) symbol** of the Huffman binary code for the fourth-order extensions of this source.

(b) Use `MATLAB` to plot the expected codeword length **per (source) symbol** of the Huffman binary code for the nth-order extensions of this source for $n = 1, 2, \ldots, 8$.

We first need to list all the probabilities for the (extended) source string (vector/block.)

Idea 1: There are $\binom{n}{k}$ source strings that has k Y and $n-k$ N.

The probability for each of these strings is $p^k(1-p)^{n-k} = 0.9^k(0.1)^{n-k}$

Ex when n = 4:

#Y

k	#	
0	1	0.0001
1	4	0.0009
2	6	0.0081
3	4	0.0729
4	1	0.6561

We then put together these value in a pmf vector using for loop.

Idea 2: Use MATLAB's "kron" command

Kronecker product

Defn. $\text{kron}(A,B) = \begin{bmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & & \ddots \end{bmatrix}$

Observe that the pmf used for the n^{th}-order source extension is the Kronecker product of
the pmf used for the 1^{st}-order extension
and
the pmf used for the $(n-1)^{\text{th}}$-order extension

(a) $\mathbb{E}[\ell(X_1 X_2 X_3 X_4)] = 1.9702$ bits per 4 source symbols

$\Rightarrow L_4 = 0.4925$ bits per source symbol.

(b)