ECS 452: In-Class Exercise # 9

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **07** / **03** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider two random variables X and Y whose joint pmf matrix is given by $\mathbf{P} = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}$. Find $I(X;Y)$.

We use the formula $I(X;Y) = H(X) + H(Y) - H(X,Y)$.

$H(X,Y)$ can be found directly from the elements in the P matrix:

$$H(X,Y) = -0.4\log_2 0.4 - 3\times 0.2\log_2 0.2 \approx 1.9219$$

$H(X)$ and $H(Y)$ can be found by first finding $p(x)$ and $q(y)$ from the P matrix:

$$P = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} \begin{matrix} \to 0.4 \\ \to 0.6 \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow \\ 0.4 & 0.6 \end{matrix}$$

$H(X) = -0.4\log_2 0.4 - 0.6\log_2 0.6 \approx 0.9710$

$H(Y) = 0.9710$

X and Y are identically distributed. So, they have the same entropy.

So, $I(X;Y) \approx 2\times 0.9710 - 1.9219 = 0.0200$

Note: You will get 0.0201 if you round to four decimal places too early.

Note: when the answer here is small, it is important that you go back and make sure that you keep enough decimal paces in your calculation.

2. Consider two random variables X and Y whose $\mathbf{p} = [0.6, 0.4]$ and $\mathbf{Q} = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}$. Find $I(X;Y)$.

First, we find the P matrix by scaling each row of the Q matrix by the corresponding $p(x)$.

$$\begin{matrix} \xrightarrow{\times 0.6} \\ \xrightarrow{\times 0.4} \end{matrix} \begin{bmatrix} 0.24 & 0.36 \\ 0.28 & 0.12 \end{bmatrix} = P$$

$$\begin{matrix} \downarrow\Sigma & \downarrow\Sigma \end{matrix}$$

$$q = [0.52 \quad 0.48]$$

Then, we follow the same entropy calculations as in question (1).

$$H(X,Y) = -0.24\log_2 0.24 - 0.28\log_2 0.28 - 0.36\log_2 0.36 - 0.12\log_2 0.12 \approx 1.9060$$

$H(X) = -0.6\log_2 0.6 - 0.4\log_2 0.4 \approx 0.9710$

$H(Y) = -0.52\log_2 0.52 - 0.48\log_2 0.48 \approx 0.9988$

Alternatively

$H(Y|X=x_1) = H([0.4 \; 0.6]) \approx 0.9710$

$H(Y|X=x_2) = H([0.7 \; 0.3]) \approx 0.8813$

$H(Y|X) = \sum_x p(x) H(Y|x) \approx 0.6\times 0.9710 + 0.4\times 0.8813 \approx 0.9351$

$I(X;Y) = H(Y) - H(Y|X) \approx 0.9988 - 0.9351 \approx 0.0638$

$$I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.9710 + 0.9988 - 1.9060 \approx 0.0638$$

Note: when the answer here is small, it is important that you go back and make sure that you keep enough decimal paces in your calculation.

3. (0 pt) Consider two random variables X and Y whose $\mathbf{Q} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}$. Find $I(X;Y)$.

Note that the two rows in Q are identical. This means $Q(y|x)$ does not depend on x. In other words, knowing the value of X does not change the (conditional) pmf of Y. Therefore, X and Y are independent which implies $I(X;Y) = 0$.

See next page for a more direct solution.

Remark: Normally, to calculate $I(X;Y)$ you will need both $\underline{p}$ and Q. So, there must be something special about Q that allows you to get $I(X;Y)$ without $\underline{p}$.

Direct calculation:

$H(Y|x) = H([0.6\ \ 0.4]) \approx 0.9710$ for any x

So, $H(Y|X) = \sum_x p(x) H(Y|x) \approx 0.9710 \underbrace{\sum_x p(x)}_{1} \approx 0.9710$

$I(X;Y) = H(Y) - H(Y|X)$. So, we need $H(Y)$ which in turn need $\underline{q}(y)$

Let's try $p(x) = \begin{cases} 1-p_0 & x=0 \\ p_0 & x=1 \\ 0, & \text{otherwise} \end{cases}$

Then,

$$\overset{\underline{p}}{[1-p\ \ \ p]} \overset{Q}{\begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}} = \overset{\underline{q}}{[0.6\ \ 0.4]} \Rightarrow H(Y) = H([0.6\ \ 0.4])$$

regardless of the value of p

same!

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.