ECS 452: In-Class Exercise # 8

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.** Only one submission is needed for each group.
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: **28** / **02** / 2019			
Name	ID (last 3 digits)		
Prapun	**5**	**5**	**5**

1. Consider two random variables X and Y whose joint pmf matrix is given by

$$Q = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{array}{l} \xleftarrow{\times 4} \\ \xleftarrow{\times 4} \\ \xleftarrow{\times 2} \end{array} \quad \mathbf{P} = \begin{array}{c|c} x \backslash y & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1/8 & 0 & 1/8 & 0 \\ 1/8 & 1/8 & 0 & 0 \\ 0 & 1/8 & 1/8 & 1/4 \end{bmatrix} \\ & \begin{array}{cccc} \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow \\ 1/4 & 1/4 & 1/4 & 1/4 \end{array} \end{array} \begin{array}{l} \xrightarrow{\Sigma} 1/4 \\ \xrightarrow{\Sigma} 1/4 \\ \xrightarrow{\Sigma} 1/2 \end{array}$$

1/8 0 1/8 0
1/8 1/8 0 0
0 1/8 1/8 1/4

Calculate the following quantities.

a. $H(X,Y) = -\sum_{(x,y)} p(x,y)\log_2 p(x,y) = -6\times\frac{1}{8}\log_2\frac{1}{8} - \frac{1}{4}\log_2\frac{1}{4}$

$= -\frac{3}{4}(-3) - \frac{1}{4}(-2) = \frac{9+2}{4} = \frac{11}{4} = 2.75$ [bits per pair]

pair of symbols (X,Y)

b. $H(X)$ First, we find p(x) by summing along each row of the P matrix.

$= -\sum_{x} p(x)\log_2 p(x) = -2\times\frac{1}{4}\log_2\frac{1}{4} - \frac{1}{2}\log_2\frac{1}{2} = -\frac{1}{2}(-2) - \frac{1}{2}(-1) = \frac{2+1}{2} = \frac{3}{2} = 1.5$

[bits per symbol]

c. $H(Y)$ First, we find q(y) by summing along each column of the P matrix.
We then know that Y is a uniform RV with four equally-likely possibilities.

Therefore, $H(Y) = \log_2 4 = 2$ [bits per symbol]

Alternatively, $H(Y) = -\sum_{y} q(y)\log_2 q(y)$
$= -\cancel{4}\times\frac{1}{\cancel{4}}\log_2\frac{1}{4}$
$= -\log_2 2^{-2} = 2$

d. $H(Y|X)$

$= H(X,Y) - H(X) = 2.75 - 1.5 = 1.25 = \frac{5}{4}$ [bits per symbol]

Note that this is calculating the (average) amount of randomness in Y (but given that we know the value of X). So the unit is per Y symbol.

e. **Q** matrix ← can be found by scaling each row of the P matrix by $\frac{1}{p(x)}$

$$\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{array}{l} \xleftarrow{\times 4} \\ \xleftarrow{\times 4} \\ \xleftarrow{\times 2} \end{array} \begin{bmatrix} 1/8 & 0 & 1/8 & 0 \\ 1/8 & 1/8 & 0 & 0 \\ 0 & 1/8 & 1/8 & 1/4 \end{bmatrix}$$

x	$p(x)$	$1/p(x)$
1	1/4	4
2	1/4	4
3	1/2	2

f. $H(Y|X=3)$

↳ We use the "x = 3" row of the Q matrix to calculate this conditional entropy.

We can also find $H(Y|X=1) = H(Y|X=2) = 1$.

$= -2\times\frac{1}{4}\log_2\frac{1}{4} - \frac{1}{2}\log_2\frac{1}{2} = -\frac{1}{2}(-2) - \frac{1}{2}(-1) = \frac{3}{2} = 1.5$ Therefore, $H(Y|X) = \sum_{x} p(x) H(Y|x)$

[bits per symbol] $= \frac{1}{4}\times 1 + \frac{1}{4}\times 1 + \frac{1}{2}\times\frac{3}{2} = \frac{5}{4}$

same as what we got in part (d).