Name ______________________ ID3 _ _ _

ECS 452: Digital Communication Systems **2017/2**

HW 7 — Due: May 11, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 4 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider a convolutional encoder whose trellis diagram is given in Figure 7.1.

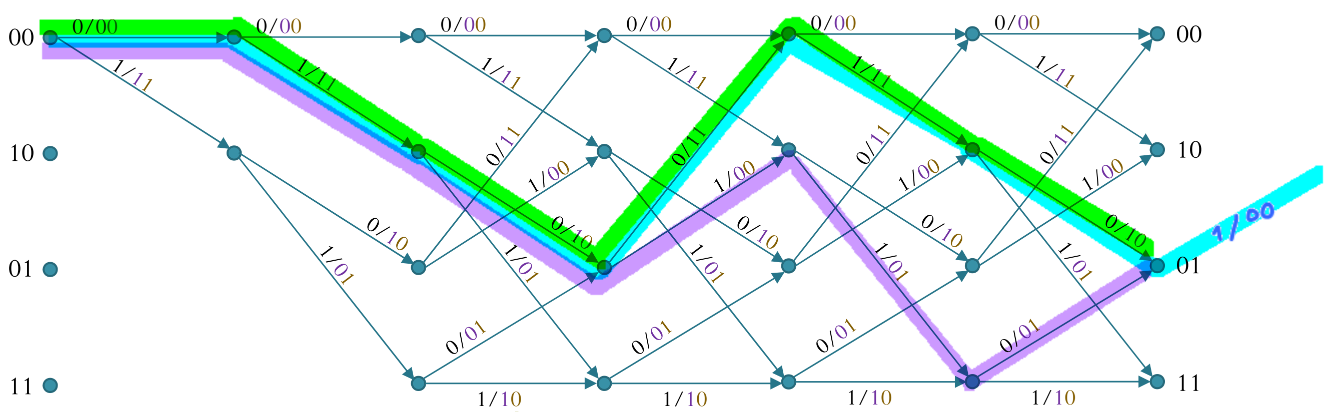

Figure 7.1: State diagram for a convolutional encoder

(a) Find the code rate $\frac{\text{one bit}}{\text{two bits}} = \frac{1}{2}$

(b) Suppose the data bits (message) are $\underline{\mathbf{b}} = [0100101]$. Find the corresponding codeword $\underline{\mathbf{x}}$.

From the blue highligthed path, we have $\underline{x}$ = [00111011111000]

(c) Find the data vector $\underline{\mathbf{b}}$ which gives the codeword $\underline{\mathbf{x}} = [001110111110]$.

From the green highlighted path, we have $\underline{b}$ = [010010]

Alternatively, because the codeword here is the same as the first 12 bits of the codeword in the previous part, we know that the data vector must be the same as the first 6 bits of the data vector from the previous part.

(d) Suppose that we observe $\underline{\mathbf{y}} = [001110000101]$ at the input of the minimum distance decoder. Explain why we can easily find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$ without applying the Viterbi algorithm.

From the purple highligthed path, we see that $\underline{y}$ itself is a codeword.
So, there is a codeword with distance = 0 from $\underline{y}$.
There can not be any codeword with smaller distance from $\underline{y}$ than 0.
So, $\hat{\underline{x}} = \underline{y}$.
From the purple highligthed path, we can also read $\hat{\underline{b}}$ = [010110]

(e) Suppose that we observe $\underline{\mathbf{y}} = [010101111110]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$. Show your work on Figure 7.2 below.

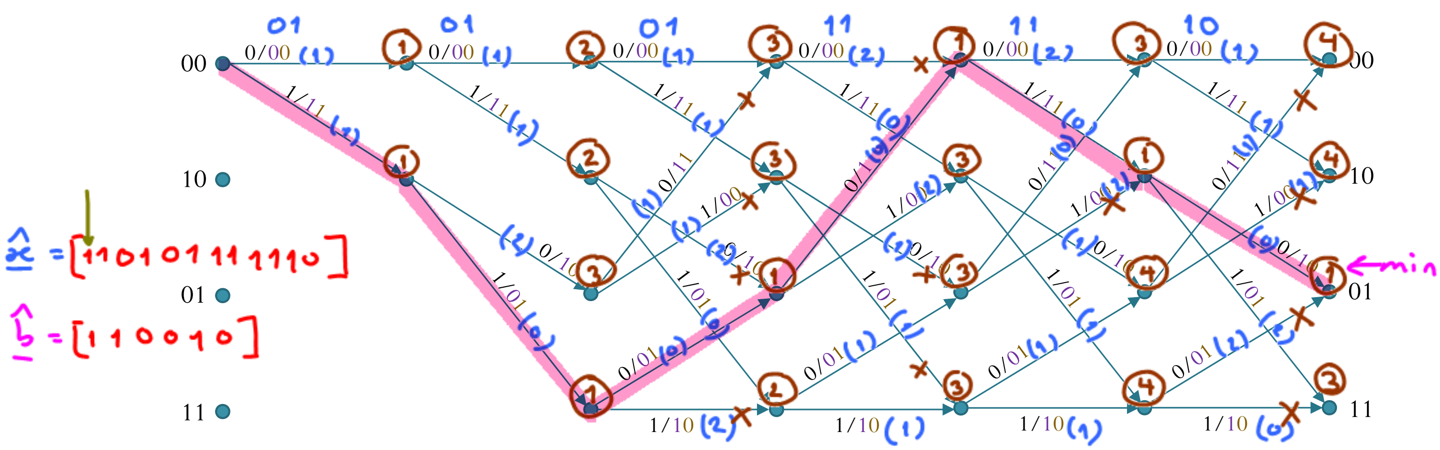

Figure 7.2: State diagram for a convolutional encoder

Make sure that all the running (cumulative) path metric are shown and the discarded branches are indicated at every steps.

$\langle \vec{v}^{(2)}, \vec{v}^{(3)} \rangle = 2+0+0+1+0 = 3$

$\langle \vec{v}^{(2)}, \vec{v}^{(4)} \rangle = 3+1+0+2+0 = 6$

Problem 2. Consider four vectors:

$\langle \vec{v}^{(1)}, \vec{v}^{(2)} \rangle = 1-1+0+0+0 = 0$

$\langle \vec{v}^{(1)}, \vec{v}^{(3)} \rangle = 2+0+1+0+1 = 4$

$\langle \vec{v}^{(1)}, \vec{v}^{(4)} \rangle = 3-1+1+0+1 = 4$

$$\mathbf{v}^{(1)} = \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} +1 \\ +1 \\ 0 \\ +1 \\ 0 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} +2 \\ 0 \\ +1 \\ +1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} +3 \\ +1 \\ +1 \\ +2 \\ -1 \end{pmatrix}.$$

(a) Use the Gram-Schmidt orthogonalization procedure (GSOP) (where the vectors are applied **in the order given**) to find the orthonormal vectors $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots$ that can be used as axes to represent $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}$, and $\mathbf{v}^{(4)}$.

1 $\vec{u}^{(1)} = \vec{v}^{(1)} = (1,-1,1,0,-1)^T$. $\Rightarrow \vec{v}^{(1)} = \vec{u}^{(1)} = \|\vec{u}^{(1)}\| \vec{e}^{(1)} = 2\vec{e}^{(1)}$

$\|\vec{u}^{(1)}\| = \sqrt{1^2+(-1)^2+1^2+0^2+(-1)^2} = \sqrt{4} = 2 \Rightarrow \vec{e}^{(1)} = \frac{\vec{u}^{(1)}}{\|\vec{u}\|^{(1)}} = \frac{1}{2}(1,-1,1,0,-1)^T$

2 $\vec{u}^{(2)} = \vec{v}^{(2)} - \underbrace{\text{proj}_{\vec{u}^{(1)}} \vec{v}^{(2)}}_{= \frac{\langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} = \frac{0}{4}\vec{u}^{(1)} = \vec{0}} = \vec{v}^{(2)} - \vec{0} = \vec{v}^{(2)} = (1,1,0,1,0)^T \Rightarrow \vec{v}^{(2)} = \vec{u}^{(2)} = \|\vec{u}^{(2)}\| \vec{e}^{(2)} = \sqrt{3}\,\vec{e}^{(2)}$

$\|\vec{u}^{(2)}\| = \sqrt{1^2+1^2+0^2+1^2+0^2} = \sqrt{3} \Rightarrow \vec{e}^{(2)} = \frac{\vec{u}^{(2)}}{\|\vec{u}^{(2)}\|} = \frac{1}{\sqrt{3}}(1,1,0,1,0)^T$

3 $\vec{u}^{(3)} = \vec{v}^{(3)} - \text{proj}_{\vec{u}^{(1)}} \vec{v}^{(3)} - \text{proj}_{\vec{u}^{(2)}} \vec{v}^{(3)}$

$\text{proj}_{\vec{u}^{(1)}} \vec{v}^{(3)} = \frac{\langle \vec{v}^{(3)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} = \frac{\langle \vec{v}^{(3)}, \vec{v}^{(1)} \rangle}{\|\vec{u}^{(1)}\|^2} \vec{u}^{(1)} = \frac{4}{4}\vec{u}^{(1)} = \vec{u}^{(1)}$

$\text{proj}_{\vec{u}^{(2)}} \vec{v}^{(3)} = \frac{\langle \vec{v}^{(3)}, \vec{u}^{(2)} \rangle}{\langle \vec{u}^{(2)}, \vec{u}^{(2)} \rangle} \vec{u}^{(2)} = \frac{\langle \vec{v}^{(3)}, \vec{v}^{(2)} \rangle}{\|\vec{u}^{(2)}\|^2} \vec{u}^{(2)} = \frac{3}{3}\vec{u}^{(2)} = \vec{u}^{(2)}$

$= \vec{v}^{(3)} - \vec{u}^{(1)} - \vec{u}^{(2)}$ $\Rightarrow \vec{v}^{(3)} = \vec{u}^{(1)} + \vec{u}^{(2)} + \cancel{\vec{u}^{(3)}}_{0} = 2\vec{e}^{(1)} + \sqrt{3}\,\vec{e}^{(2)}$

$= (2,0,1,1,-1)^T - (1,-1,1,0,-1)^T - (1,1,0,1,0)^T$

$= \vec{0}$ $\Rightarrow$ discarded.

No $\vec{u}^{(3)}$ because it is discarded in the previous step.

④ $\vec{u}^{(4)} = \vec{v}^{(4)} - \underbrace{\operatorname{proj}_{\vec{u}^{(1)}} \vec{v}^{(4)}}_{} - \underbrace{\operatorname{proj}_{\vec{u}^{(2)}} \vec{v}^{(4)}}_{}$

$$= \frac{\langle \vec{v}^{(4)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} \qquad = \frac{\langle \vec{v}^{(4)}, \vec{u}^{(2)} \rangle}{\langle \vec{u}^{(2)}, \vec{u}^{(2)} \rangle} \vec{u}^{(2)}$$

$$= \frac{\langle \vec{v}^{(4)}, \vec{v}^{(1)} \rangle}{\|\vec{u}^{(1)}\|^2} \vec{u}^{(1)} \qquad = \frac{\langle \vec{v}^{(4)}, \vec{u}^{(2)} \rangle}{\|\vec{u}^{(2)}\|^2} \vec{u}^{(2)}$$

$$= \frac{4}{4} \vec{u}^{(1)} = \vec{u}^{(1)} \qquad = \frac{6}{3} \vec{u}^{(2)} = 2\vec{u}^{(2)}$$

$= \vec{v}^{(4)} - \vec{u}^{(1)} - 2\vec{u}^{(2)} \Rightarrow \vec{v}^{(4)} = \vec{u}^{(1)} + 2\vec{u}^{(2)} + \cancel{\vec{u}^{(4)}}\ 0 = 2\vec{e}^{(1)} + 2\sqrt{3}\,\vec{e}^{(2)}$

$= (3,1,1,2,-1)^T - (1,-1,1,0,-1)^T - 2(1,1,0,1,0)^T$

$= \vec{0} \Rightarrow$ discarded.

Therefore, we only need two axes (orthonormal vectors) to represent the four vectors.

$\hookrightarrow \vec{e}^{(1)} = \frac{1}{2}(1,-1,1,0,-1)^T$, $\vec{e}^{(2)} = \frac{1}{\sqrt{3}}(1,1,0,1,0)^T$

(b) Find the corresponding vectors $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, $\mathbf{c}^{(3)}$, and $\mathbf{c}^{(4)}$ that represent $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$, and $\mathbf{v}^{(4)}$ in the new axes derived in the previous part.

$\vec{v}^{(1)} = \vec{u}^{(1)} = 2\vec{e}^{(1)} + 0\vec{e}^{(2)} \Rightarrow \vec{c}^{(1)} = (2,0)^T$

$\vec{v}^{(2)} = \vec{u}^{(2)} = 0\vec{e}^{(1)} + \sqrt{3}\,\vec{e}^{(2)} \Rightarrow \vec{c}^{(2)} = (0,\sqrt{3})^T$

$\vec{v}^{(3)} = \vec{u}^{(1)} + \vec{u}^{(2)} = 2\vec{e}^{(1)} + \sqrt{3}\,\vec{e}^{(2)} \Rightarrow \vec{c}^{(3)} = (2,\sqrt{3})^T$

$\vec{v}^{(4)} = \vec{u}^{(1)} + 2\vec{u}^{(2)} = 2\vec{e}^{(1)} + 2\sqrt{3}\,\vec{e}^{(2)} \Rightarrow \vec{c}^{(4)} = (2, 2\sqrt{3})^T$

Extra Question

Here is an optional question for those who want more practice.

Problem 3. Consider a convolutional code generated by the encoder shown in Figure 7.3. Suppose that we observe $\underline{\mathbf{y}} = [110111000110]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$. Caution: The trellis diagram is not the same as the one used in Problem 1.

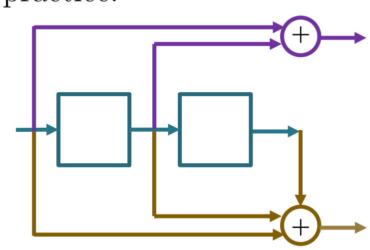

Figure 7.3: Encoder for Problem 3

Solution: Observe that the circuit diagram is exactly the same as the one used in Exercise 15. We have already found its state diagram; this is shown in Figure 7.4a. From the state diagram, we can then create the code trellis as shown in Figure 7.4b.

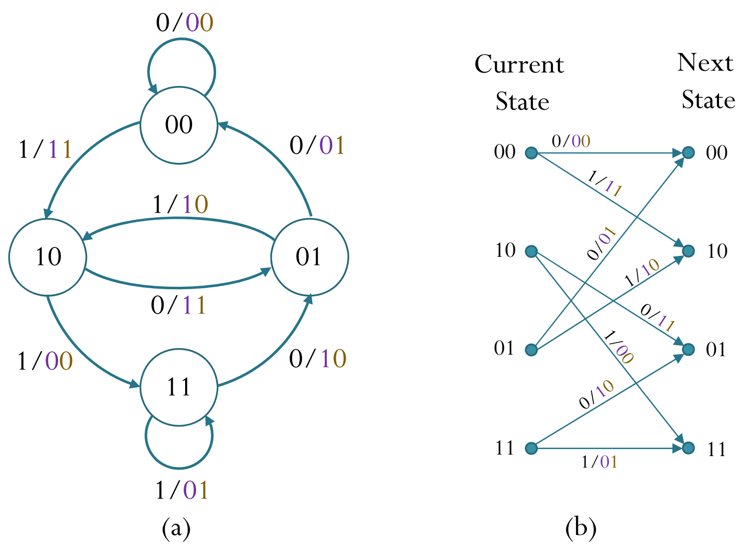

Figure 7.4: (a) State diagram and (b) Trellis diagram for Problem 3

For Viterbi decoding, the trellis diagram is shown in Figure 7.5. (Don't forget that the trellis diagram always starts with the all-zero state.)

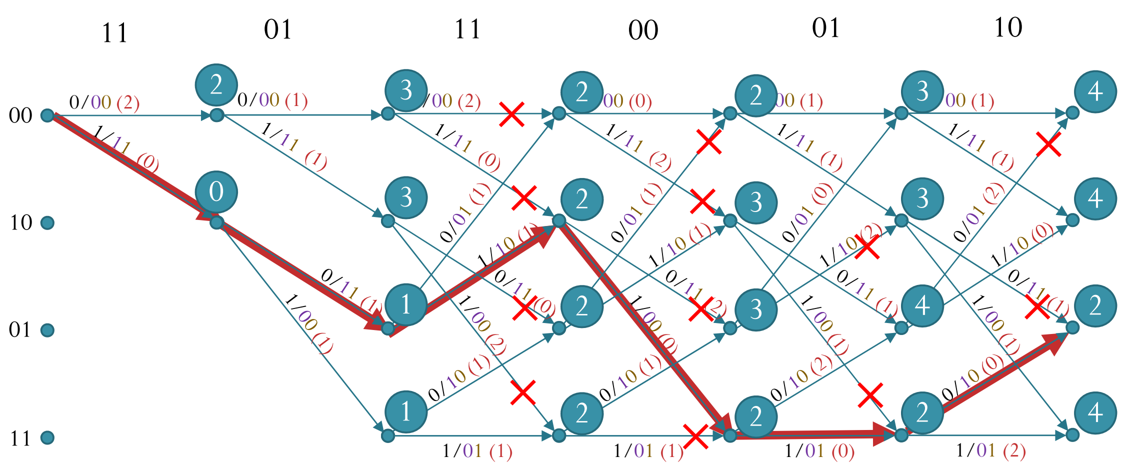

Figure 7.5: Trellis diagram for Problem 3.

Tracing back the trellis diagram, we get $\boxed{\hat{\underline{\mathbf{b}}} = [101110]}$ and $\boxed{\hat{\underline{\mathbf{x}}} = [111110000110]}$.