ECS 452: Digital Communication Systems **2016/2**

HW 8 — Due: May 16

Lecturer: Asst. Prof. Dr. Prapun Suksompong

Instructions

(a) This assignment has 4 pages.

(b) (1 pt) Write your first name and the last three digit of your student ID on the upper-right corner of ***every*** submitted sheet.

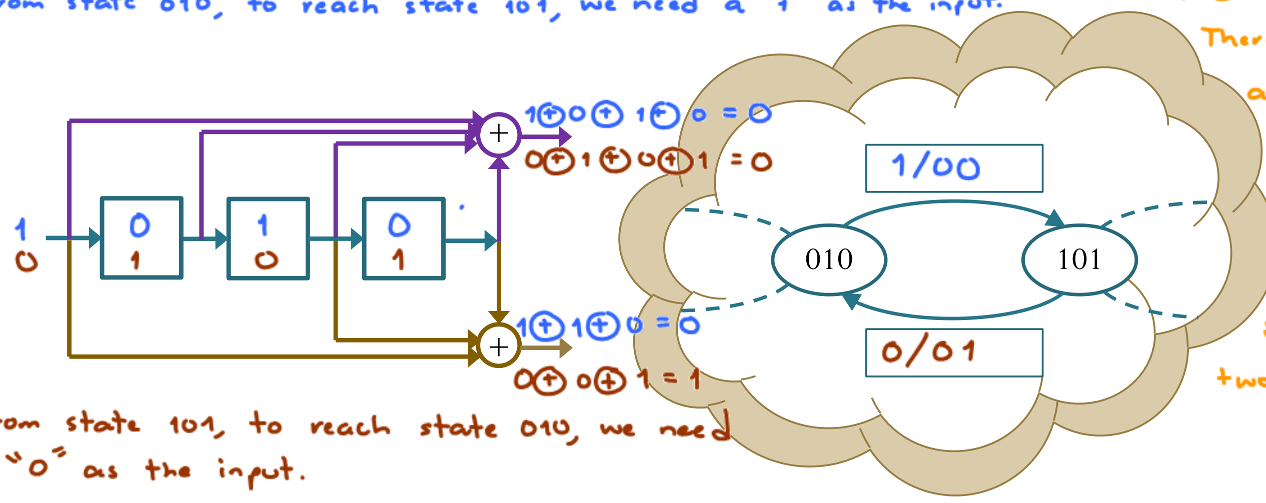
(c) (1 pt) For each part, write your explanation/derivation and answer in the space provided.

(d) (8 pt) It is important that you try to solve all non-optional problems.

(e) Late submission will be heavily penalized.

Problem 1. Consider a convolutional encoder whose circuit diagram and a part of the corresponding state diagram is given in Figure 8.1. Write the suitable labels for the two arrows shown in the state diagram.

From state 010, to reach state 101, we need a "1" as the input.



There are 3 FFs. Therefore, there are $2^3 = 8$ possible states. However, here, we show only two of them.

From state 101, to reach state 010, we need a "0" as the input.

Figure 8.1: Circuit diagram and a part of the corresponding state diagram for a convolutional encoder. Only two states are shown in the state diagram.

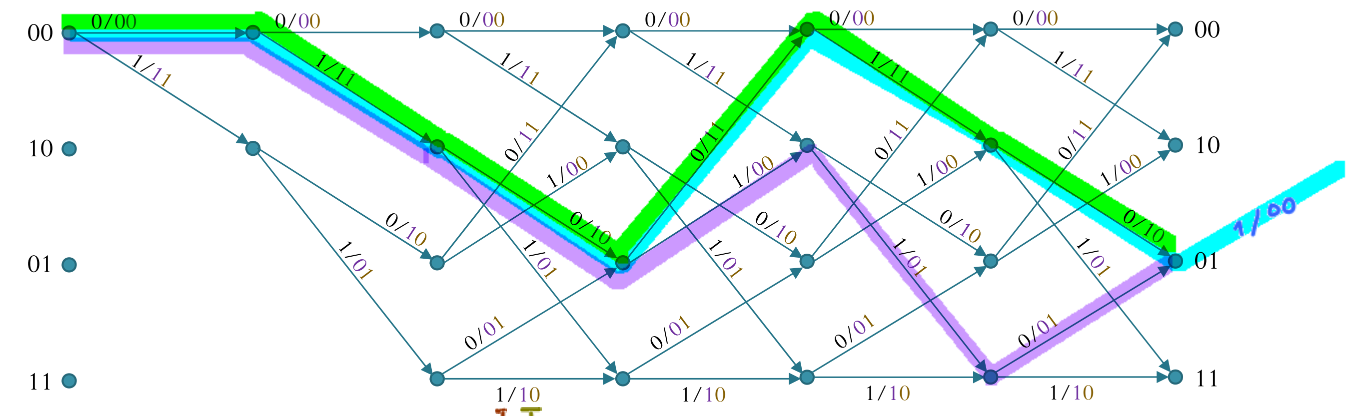


Figure 8.2: State diagram for a convolutional encoder

Problem 2. Consider a convolutional encoder whose trellis diagram is given in Figure 8.2.

(a) Find the code rate $\frac{\text{one bit}}{\text{two bits}} = \frac{1}{2}$

(b) Suppose the data bits (message) are $\underline{\mathbf{b}} = [0100101]$. Find the corresponding codeword $\underline{\mathbf{x}}$.

From the blue highligthed path, we have $\underline{x}$ = [00111011111000]

(c) Find the data vector $\underline{\mathbf{b}}$ which gives the codeword $\underline{\mathbf{x}} = [001110111110]$.

From the green highlighted path, we have $\underline{b}$ = [010010]

Alternatively, because the codeword here is the same as the first 12 bits of the codeword in the previous part, we know that the data vector must be the same as the first 6 bits of the data vector from the previous part.

(d) Suppose that we observe $\underline{\mathbf{y}} = [001110000101]$ at the input of the minimum distance decoder. Explain why we can easily find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$ without applying the Viterbi algorithm.

From the purple highligthed path, we see that $\underline{y}$ itself is a codeword.
So, there is a codeword ($\underline{y}$) with distance = 0 from $\underline{y}$.
There can not be any codeword with smaller distance from $\underline{y}$ than 0.
So, $\hat{\underline{x}} = \underline{y}$.
From the purple highligthed path, we can also read $\hat{\underline{b}}$ = [010110]

(e) Suppose that we observe $\underline{\mathbf{y}} = [01\,01\,01\,11\,11\,10]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$. Show your work on Figure 8.3 below.

$\hat{\underline{x}} = [110101111110]$

$\hat{\underline{b}} = [110010]$

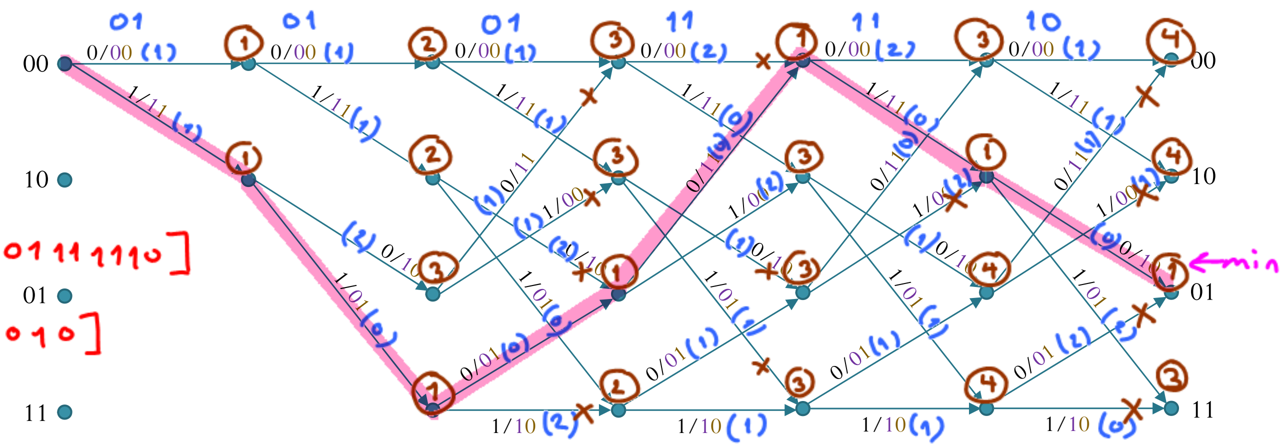


Figure 8.3: State diagram for a convolutional encoder

Make sure that all the running (cumulative) path metric are shown and the discarded branches are indicated at every steps.

Problem 3. Consider four vectors:

$\langle \vec{v}^{(1)}, \vec{v}^{(2)} \rangle = 1-1+0+0+0 = 0$

$\langle \vec{v}^{(1)}, \vec{v}^{(3)} \rangle = 2+0+1+0+1 = 4$

$\langle \vec{v}^{(1)}, \vec{v}^{(4)} \rangle = 3-1+1+0+1 = 4$

$\langle \vec{v}^{(2)}, \vec{v}^{(3)} \rangle = 2+0+0+1+0 = 3$

$\langle \vec{v}^{(2)}, \vec{v}^{(4)} \rangle = 3+1+0+2+0 = 6$

$$\mathbf{v}^{(1)} = \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} +1 \\ +1 \\ 0 \\ +1 \\ 0 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} +2 \\ 0 \\ +1 \\ +1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} +3 \\ +1 \\ +1 \\ +2 \\ -1 \end{pmatrix}.$$

(a) Use the Gram-Schmidt orthogonalization procedure (GSOP) (where the vectors are applied **in the order given**) to find the orthonormal vectors $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots$ that can be used as axes to represent $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}$, and $\mathbf{v}^{(4)}$.

① $\vec{u}^{(1)} = \vec{v}^{(1)} = (1,-1,1,0,-1)^T$. $\Rightarrow \vec{v}^{(1)} = \vec{u}^{(1)} = \|\vec{u}^{(1)}\| \vec{e}^{(1)} = 2\vec{e}^{(1)}$

$\|\vec{u}^{(1)}\| = \sqrt{1^2+(-1)^2+1^2+0^2+(-1)^2} = \sqrt{4} = 2 \Rightarrow \vec{e}^{(1)} = \dfrac{\vec{u}^{(1)}}{\|\vec{u}\|^{(1)}} = \dfrac{1}{2}(1,-1,1,0,-1)^T$

② $\vec{u}^{(2)} = \vec{v}^{(2)} - \underbrace{\text{proj}_{\vec{u}^{(1)}} \vec{v}^{(2)}}_{= \frac{\langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} = \frac{0}{4} \vec{u}^{(1)} = \vec{0}} = \vec{v}^{(2)} - \vec{0} = \vec{v}^{(2)} = (1,1,0,1,0)^T \Rightarrow \vec{v}^{(2)} = \vec{u}^{(2)} = \|\vec{u}^{(2)}\| \vec{e}^{(2)} = \sqrt{3}\,\vec{e}^{(2)}$

$\|\vec{u}^{(2)}\| = \sqrt{1^2+1^2+0^2+1^2+0^2} = \sqrt{3} \Rightarrow \vec{e}^{(2)} = \dfrac{\vec{u}^{(2)}}{\|\vec{u}^{(2)}\|} = \dfrac{1}{\sqrt{3}}(1,1,0,1,0)^T$

(3) $$\vec{u}^{(3)} = \vec{v}^{(3)} - \underbrace{\text{proj}_{\vec{u}^{(1)}} \vec{v}^{(3)}} - \underbrace{\text{proj}_{\vec{u}^{(2)}} \vec{v}^{(3)}}$$

$$\text{proj}_{\vec{u}^{(1)}} \vec{v}^{(3)} = \frac{\langle \vec{v}^{(3)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} = \frac{\langle \vec{v}^{(3)}, \vec{v}^{(1)} \rangle}{\|\vec{u}^{(1)}\|^2} \vec{u}^{(1)} = \frac{4}{4} \vec{u}^{(1)} = \vec{u}^{(1)}$$

$$\text{proj}_{\vec{u}^{(2)}} \vec{v}^{(3)} = \frac{\langle \vec{v}^{(3)}, \vec{u}^{(2)} \rangle}{\langle \vec{u}^{(2)}, \vec{u}^{(2)} \rangle} \vec{u}^{(2)} = \frac{\langle \vec{v}^{(3)}, \vec{v}^{(2)} \rangle}{\|\vec{u}^{(2)}\|^2} \vec{u}^{(2)} = \frac{3}{3} \vec{u}^{(2)} = \vec{u}^{(2)}$$

$$= \vec{v}^{(3)} - \vec{u}^{(1)} - \vec{u}^{(2)} \quad \Rightarrow \quad \vec{v}^{(3)} = \vec{u}^{(1)} + \vec{u}^{(2)} + \cancel{\vec{u}^{(3)}}_{0} = 2\vec{e}^{(1)} + \sqrt{3}\,\vec{e}^{(2)}$$

$$= (2,0,1,1,-1)^T - (1,-1,1,0,-1)^T - (1,1,0,1,0)^T$$

$$= \vec{0} \Rightarrow \text{discarded.}$$

No $\vec{u}^{(3)}$ because it is discarded in the previous step.

(4) $$\vec{u}^{(4)} = \vec{v}^{(4)} - \underbrace{\text{proj}_{\vec{u}^{(1)}} \vec{v}^{(4)}} - \underbrace{\text{proj}_{\vec{u}^{(2)}} \vec{v}^{(4)}}$$

$$\text{proj}_{\vec{u}^{(1)}} \vec{v}^{(4)} = \frac{\langle \vec{v}^{(4)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} = \frac{\langle \vec{v}^{(4)}, \vec{v}^{(1)} \rangle}{\|\vec{u}^{(1)}\|^2} \vec{u}^{(1)} = \frac{4}{4} \vec{u}^{(1)} = \vec{u}^{(1)}$$

$$\text{proj}_{\vec{u}^{(2)}} \vec{v}^{(4)} = \frac{\langle \vec{v}^{(4)}, \vec{u}^{(2)} \rangle}{\langle \vec{u}^{(2)}, \vec{u}^{(2)} \rangle} \vec{u}^{(2)} = \frac{\langle \vec{v}^{(4)}, \vec{u}^{(2)} \rangle}{\|\vec{u}^{(2)}\|^2} \vec{u}^{(2)} = \frac{6}{3} \vec{u}^{(2)} = 2\vec{u}^{(2)}$$

$$= \vec{v}^{(4)} - \vec{u}^{(1)} - 2\vec{u}^{(2)} \quad \Rightarrow \quad \vec{v}^{(4)} = \vec{u}^{(1)} + 2\vec{u}^{(2)} + \cancel{\vec{u}^{(4)}}_{0} = 2\vec{e}^{(1)} + 2\sqrt{3}\,\vec{e}^{(2)}$$

$$= (3,1,1,2,-1)^T - (1,-1,1,0,-1)^T - 2(1,1,0,1,0)^T$$

$$= \vec{0} \Rightarrow \text{discarded.}$$

Therefore, we only need two axes (orthonormal vectors) to represent the four vectors.

↳ $\vec{e}^{(1)} = \frac{1}{2}(1,-1,1,0,-1)^T$, $\vec{e}^{(2)} = \frac{1}{\sqrt{3}}(1,1,0,1,0)^T$

(b) Find the corresponding vectors $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, $\mathbf{c}^{(3)}$, and $\mathbf{c}^{(4)}$ that represent $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$, and $\mathbf{v}^{(4)}$ in the new axes derived in the previous part.

$$\vec{v}^{(1)} = \vec{u}^{(1)} = 2\vec{e}^{(1)} + 0\vec{e}^{(2)} \Rightarrow \vec{c}^{(1)} = (2,0)^T$$

$$\vec{v}^{(2)} = \vec{u}^{(2)} = 0\vec{e}^{(1)} + \sqrt{3}\,\vec{e}^{(2)} \Rightarrow \vec{c}^{(2)} = (0,\sqrt{3})^T$$

$$\vec{v}^{(3)} = \vec{u}^{(1)} + \vec{u}^{(2)} = 2\vec{e}^{(1)} + \sqrt{3}\,\vec{e}^{(2)} \Rightarrow \vec{c}^{(3)} = (2,\sqrt{3})^T$$

$$\vec{v}^{(4)} = \vec{u}^{(1)} + 2\vec{u}^{(2)} = 2\vec{e}^{(1)} + 2\sqrt{3}\,\vec{e}^{(2)} \Rightarrow \vec{c}^{(4)} = (2, 2\sqrt{3})^T$$