ECS 452: Digital Communication Systems	2016/2

HW 2 — Due: Feb 21

Lecturer: Asst. Prof. Dr. Prapun Suksompong

Instructions

(a) This assignment has 4 pages.

(b) (1 pt) Write your first name and the last three digit of your student ID on the upper-right corner of ***every*** submitted sheet.

(c) (1 pt) For each part, write your explanation/derivation and answer in the space provided.

(d) (8 pt) It is important that you try to solve all non-optional problems.

(e) Late submission will be heavily penalized.

Problem 1. Consider a random variable X whose support is $S_X = \{x_1, x_2, \dots, x_n\}$. Let $p_k = p_X(x_k)$. Then, the entropy of X is

$$H(X) = -\sum_{k=1}^{n} p_X(x_k) \log_2 p_X(x_k) = -\sum_{k=1}^{n} p_k (\log_2 p_k).$$

As discussed in class, observe that the entropy of X does not depend on the specific values $x_1, x_2, \dots, x_n$ in its support. The entropy is a function of the probability values $p_1, p_2, \dots, p_n$ in the pmf. Therefore, to calculate entropy, it is enough to specify these probabilities. From this observation, we may write the entropy as a function $H(\underline{\mathbf{p}})$ of a probability vector $\underline{\mathbf{p}}$ constructed from (positive probabilities in) the pmf.

In general, given a probability vector $\underline{\mathbf{p}} = [p_1, p_2, \dots, p_n]$ whose elements are nonnegative and sum to one, we calculate the corresponding entropy value by

$$H(\underline{\mathbf{p}}) = -\sum_{k=1}^{n} p_k (\log_2 p_k).$$

In each row of the following table, compare the entropy value in the first column with the entropy value in the third column by writing ">", "=", or "<" in the second column. Watch out for approximation and round-off error.

$H(\underline{\mathbf{p}})$ when $\underline{\mathbf{p}} = [0.3, 0.7]$.		$H(\underline{\mathbf{q}})$ when $\underline{\mathbf{q}} = [0.8, 0.2]$.
$H(\underline{\mathbf{p}})$ when $\underline{\mathbf{p}} = [0.3, 0.3, 0.4]$.		$H(\underline{\mathbf{q}})$ when $\underline{\mathbf{q}} = [0.4, 0.3, 0.3]$.
$H(X)$ when $p(x) = \begin{cases} 0.3, & x \in \{1,2\}, \\ 0.2, & x \in \{3,4\}, \\ 0, & \text{otherwise.} \end{cases}$		$H(\underline{\mathbf{q}})$ when $\underline{\mathbf{q}} = [0.4, 0.3, 0.3]$.

Problem 2. A memoryless source emits two possible message Y(es) and N(o) with probability 0.9 and 0.1, respectively.

(a) Determine the entropy (per source symbol) of this source.

(b) Find the expected codeword length per symbol of the Huffman binary code for the third-order extensions of this source.

(c) Use `MATLAB` to find the expected codeword length **per (source) symbol** of the Huffman binary code for the fourth-order extensions of this source.

(i) Put your answer here.

(ii) Attach the **printout** (not handwritten) of your `MATLAB` script (**highlighting** the modified parts if you start from the provided class example) and the expressions/results displayed in the command window.

(d) Use `MATLAB` to plot the expected codeword length **per (source) symbol** of the Huffman binary code for the nth-order extensions of this source for $n = 1, 2, \ldots, 8$. Attach the **printout** of your plot.

Problem 3. Consider a BSC whose crossover probability for each bit is $p = 0.35$. Suppose $P[X = 0] = 0.45$.

(a) Draw the channel diagram.

(b) Find the channel matrix $\mathbf{Q}$.

(c) Find the joint pmf matrix $\mathbf{P}$.

(d) Find the row vector $\underline{\mathbf{q}}$ which contains the pmf of the channel output Y.

Problem 4. Consider a DMC whose $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{1, 2, 3\}$, and $\mathbf{Q} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$.

Suppose the input probability vector is $\underline{\mathbf{p}} = [0.2, 0.4, 0.4]$.

(a) Find the joint pmf matrix $\mathbf{P}$.

(b) Find the row vector $\underline{\mathbf{q}}$ which contains the pmf of the channel output Y.

Problem 5. Consider a BAC whose $Q(1|0) = 0.35$ and $Q(0|1) = 0.55$. Suppose $P[X = 0] = 0.4$.

(a) Draw the channel diagram.

(b) Find the joint pmf matrix $\mathbf{P}$.

(c) Find the row vector $\underline{\mathbf{q}}$ which contains the pmf of the channel output Y.

Extra Question

Here is an optional question for those who want more practice.

Problem 6. ***Optimal code lengths that require one bit above entropy***: The source coding theorem says that the Huffman code for a random variable X has an expected length strictly less than $H(X) + 1$. Give an example of a random variable for which the expected length of the Huffman code (without any source extension) is very close to $H(X) + 1$.