### Instructions

1. Separate into groups of no more than three persons.
2. The group cannot be the same as your former group.
3. Only one submission is needed for each group.
4. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
5. Do not panic.

1. Consider a random variable $X$ having four possible values. Their probabilities are

   \[
   \frac{1}{6}, \frac{2}{9}, \frac{5}{18}, \frac{1}{3}.
   \]

   Find the **expected codeword length** (per symbol) when **Huffman coding** is used (without extension) to encode an i.i.d sequence generated by this random variable.

   ![Huffman Tree](image)

   **Recipe:**
   - Combine the two least likely (combined) symbols

   \[
   \mathbb{E}[\ell(X)] = 2 \text{ bits (per symbol)}
   \]

2. Consider a random variable $X$ having four possible values. Their probabilities are

   \[
   \frac{1}{8}, \frac{5}{24}, \frac{7}{24}, \frac{3}{8}.
   \]

   Find the **expected codeword length** (per symbol) when **Huffman coding** is used (without extension) to encode an i.i.d sequence generated by this random variable.

   ![Huffman Tree](image)

   **Recipe:**
   - Combine the two least likely (combined) symbols

   \[
   \mathbb{E}[\ell(X)] \approx 1.9593 \text{ bits}
   \]
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1. Write each of the following quantities in the form X.XXX (possibly with the help of your calculator).

   a. \(-\log_2(1/16) = -\log_2\left(\frac{1}{16}\right) = -\log_2 2^{-5} = -(-5) = 5 \approx 3.000\)

   b. \(-\log_2(0.3) \approx 1.7370\)

   c. \(-(0.3)\log_2(0.3) -(0.7)\log_2(0.7) \approx 0.8813\)

2. Consider a random variable X having four possible values. Their probabilities are

   \[ p = \left[ \frac{1}{6}, \frac{2}{9}, \frac{5}{18}, \frac{1}{3} \right] \]

   Find the entropy (per symbol) of this random variable.

   \[
   H(p) = H(X) = -\sum_{x} p_x(x) \log_2 p_x(x) \\
   = -\frac{1}{6} \log_2 \frac{1}{6} - \frac{2}{9} \log_2 \frac{2}{9} - \frac{5}{18} \log_2 \frac{5}{18} - \frac{1}{3} \log_2 \frac{1}{3} \\
   \approx 0.4308 + 0.4822 + 0.5133 + 0.5283 \approx 1.9547
   \]

3. Consider a random variable X having four possible values. Their probabilities are

   \[ p = \left[ \frac{1}{8}, \frac{5}{24}, \frac{7}{24}, \frac{3}{8} \right] \]

   Find the entropy (per symbol) of this random variable.

   \[
   H(p) = H(X) = -\sum_{x} p_x(x) \log_2 p_x(x) \\
   = -\frac{3}{24} \log_2 \frac{3}{24} - \frac{5}{24} \log_2 \frac{5}{24} - \frac{7}{24} \log_2 \frac{7}{24} - \frac{9}{24} \log_2 \frac{9}{24} \\
   \approx 0.3750 + 0.4715 + 0.5175 + 0.5306 = 1.9756
   \]
**Solution**

**ECS 452: In-Class Exercise # 3**

**Instructions**

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1. No need to provide any explanation for this question.

Consider a DMC whose samples of input and output are provided below:

\[
x: \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\
y: \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1
\]

Estimate the following quantities:

- **a.** \( X = \{0, 1\} \)
- **b.** \( P[X = 0] \approx \frac{3}{15} = \frac{1}{5} = 0.2 \)
- **c.** \( P[1] = P[X = 1] \approx \frac{12}{15} = \frac{4}{5} = 0.8 \)
- **d.** \( P[0] = P[Y = 0] \approx \frac{2}{15} \approx 0.133 \)
- **e.** \( P = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \)
- **f.** \( q[1] = P[Y = 1] \approx \frac{13}{15} \approx 0.867 \)
- **g.** \( P[Y = 0 | X = 0] \approx \frac{2}{3} \approx 0.667 \)
- **h.** \( P[Y|X=1|0] = P[Y = 1 | X = 0] \approx \frac{1}{3} \approx 0.333 \)
- **i.** \( Q[0|1] = P[Y = 0 | X = 1] \approx \frac{5}{12} = 0 \)
- **j.** \( Q[1|1] = P[Y = 1 | X = 1] = 1 \)
- **k.** Matrix \( Q \approx \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \)
- **l.** \( P[X = 0, Y = 0] \approx \frac{2}{15} \leftarrow \text{Note that this is the same as} \right. \)

\[
P[Y = 0 | X = 0] P[X = 0] = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15} \]
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1. Consider a DMC whose $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{1, 2, 3, 4\}$, and $Q = \begin{bmatrix} 0.2 & 0.6 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.3 & 0.1 \end{bmatrix}$.

Suppose the input probability vector is $p = [0.2 \ 0.1 \ 0.7]$.

a. Find the output probability vector $q$

$$q = pQ = \begin{bmatrix} 0.2 & 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.26 & 0.4 & 0.24 & 0.1 \end{bmatrix}$$

b. Find the joint pmf matrix $P$.

Multiply each row in the $Q$ matrix by its corresponding $p(x)$

$$Q = \begin{bmatrix} 0.2 & 0.4 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.04 & 0.12 & 0.02 & 0.02 \\ 0.01 & 0.07 & 0.04 & 0.04 \\ 0.21 & 0.21 & 0.21 & 0.07 \end{bmatrix} = P$$

$$q = \begin{bmatrix} 0.26 & 0.4 & 0.24 & 0.1 \end{bmatrix}$$

c. Suppose the naïve decoder is used. Find the corresponding $P(E)$.

$$P = \begin{bmatrix} 0.04 & 0.12 & 0.02 & 0.02 \\ 0.01 & 0.03 & 0.04 & 0.04 \\ 0.21 & 0.21 & 0.21 & 0.07 \end{bmatrix}$$

$$P(E) = 0.04 + 0.07 + 0.21 = 0.32$$

$$P(E) = 1 - 0.32 = 0.68$$
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5. Do not panic.

1. Consider a DMC whose $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{1, 2, 3, 4\}$, and $Q = \begin{bmatrix} 0.2 & 0.6 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.3 & 0.1 \end{bmatrix}$.

Suppose the input probability vector is $p = [0.2, 0.1, 0.7]$. 

a. Suppose the following decoder is used. Find the corresponding $P(\mathcal{E})$.

\[
P = \begin{bmatrix} 0.04 & 0.12 & 0.02 & 0.02 \\ 0.01 & 0.07 & 0.01 & 0.01 \\ 0.21 & 0.21 & 0.21 & 0.07 \end{bmatrix}
\]

\[
P(\mathcal{C}) = 0.04 + 0.12 + 0.01 + 0.07 = 0.24
\]

\[
P(\mathcal{E}) = 1 - 0.24 = 0.76
\]

Even worse than the naive decoder.

b. Find the MAP detector and its error probability.

\[
P(\mathcal{C}) = 0.21 + 0.21 + 0.21 + 0.07 = 0.7
\]

\[
P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - 0.7 = 0.3
\]

c. Find the ML detector and its error probability.

\[
P(\mathcal{C}) = 0.21 + 0.07 + 0.21 + 0.02 = 0.51
\]

\[
P(\mathcal{E}) = 1 - 0.51 = 0.49
\]
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1. Consider two random variables $X$ and $Y$ whose joint pmf matrix is given by

$$P = \begin{bmatrix}
1 & \frac{3}{18} & \frac{5}{18} \\
2 & \frac{4}{9} & \frac{2}{9}
\end{bmatrix}$$

Calculate the following quantities. Except for part (e), all of your answers should be of the form $X.XXXX$.

a. $H(X, Y) = -\sum \frac{1}{18} \log_2 \frac{1}{18} - \frac{5}{18} \log_2 \frac{5}{18} - \frac{4}{9} \log_2 \frac{4}{9} - \frac{2}{9} \log_2 \frac{2}{9}$

$\simeq 1.7472$ bits (per symbol)

b. $H(X) = H([\frac{1}{3} \ \frac{2}{3}]) = -\frac{1}{3} \log_2 \frac{1}{3} - \frac{2}{3} \log_2 \frac{2}{3} \simeq 0.9183$ bits (per symbol)

c. $H(Y) = H([\frac{1}{2} \ \frac{1}{2}]) = \log_2 2 = 1.000$ bit (per symbol)

For uniform RV, the entropy is simply $\log_2$ of the size of its support. Alternatively,

$H(Y|X) = H(X, Y) - H(X) \approx 1.7472 - 0.9183 \simeq 0.8289$ bits (per symbol)

d. $H(Y|X) = H(X, Y) - H(X) \approx 1.7472 - 0.9183 \simeq 0.8289$ bits (per symbol)

e. Q matrix

$$P = \begin{bmatrix}
\frac{1}{18} & \frac{5}{18} \\
\frac{4}{9} & \frac{2}{9}
\end{bmatrix}$$

$H(Y|1) = H([\frac{1}{6} \ \frac{5}{6}]) \approx 0.6500$ bits (per symbol)

$H(Y|2) = H([\frac{3}{5} \ \frac{2}{5}]) \approx 0.9183$ bits (per symbol)
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Consider two random variables $X$ and $Y$ whose joint pmf matrix is given by

$$ P = \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}. $$

Find $I(X;Y)$.

Note: when the value of $I(X;Y)$ is this small when compared with the entropy values, you should be careful with the rounding error.

Consider two random variables $X$ and $Y$ whose pmf is given by

$$ p = \begin{bmatrix} 1/5 & 4/5 \\ 2/5 & 3/5 \end{bmatrix} $$

and

$$ q = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}. $$

Find $I(X;Y)$.

Consider two random variables $X$ and $Y$ whose pmf is similar to the previous case. Find $I(X;Y)$.

Note that the two rows in $Q$ are identical. This means $Q(y|x)$ does not depend on $x$. In other words, knowing the value of $X$ does not change the (conditional) pmf of $Y$. Therefore, $X$ and $Y$ are independent which implies $I(X;Y) = 0$.

See next page for a more direct solution.
Remark: Normally, to calculate $I(X;Y)$ you will need both $p_e$ and $Q$.

So, there must be something special about $Q$ that allows you to get $I(X;Y)$ without $p_e$.

**Direct calculation:**

\[
H(Y|X) = H\left(\left[\frac{3}{5}, \frac{2}{5}\right]\right) = 0.7219 \quad \text{for any } \alpha.
\]

So,
\[
H(Y|X) = \sum_x p_x(x) H(Y|X) = 0.7219 \Rightarrow p(X|e) = 0.7219.
\]

\[
I(X;Y) = H(Y) - H(Y|X). \quad \text{So, we need } H(Y) \text{ which in turn need } p_Y.
\]

Let's try
\[
p_x(e) = \begin{cases} \frac{1-p_e}{2} & \alpha = 0 \\ \frac{p_e}{2} & \alpha = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Then,
\[
\begin{bmatrix} 1-p & p \\ p_y & 0 \end{bmatrix} \begin{bmatrix} y_3 & y_\bar{3} \\ y_5 & y_\bar{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \end{bmatrix} \Rightarrow H(Y) = H\left(\left[\frac{3}{5}, \frac{2}{5}\right]\right) = H(Y|X)
\]

regardless of the value of $p_e$.

Therefore,
\[
I(X;Y) = H(Y) - H(Y|X) = 0.
\]
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1. For each of the following DMC's probability transition matrices $Q$, (i) indicate whether the corresponding DMC is weakly symmetric (Yes or No), (ii) evaluate the corresponding capacity value (your answer should be of the form $X.XXXX$), and (iii) specify the channel input pmf (a row vector $p$) that achieves the capacity.

<table>
<thead>
<tr>
<th>$Q$</th>
<th>Weakly Symmetric?</th>
<th>$C$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} 1/6 &amp; 1/3 &amp; 1/2 \ 1/2 &amp; 1/6 &amp; 1/3 \ 1/3 &amp; 1/2 &amp; 1/6 \end{bmatrix}$</td>
<td>Yes. BSC is symmetric and hence weakly symmetric.</td>
<td>$C = \log_2 3 \approx 1.6605$</td>
<td>$P^* = [1/3, 1/3, 1/3]$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 0 &amp; 1/2 &amp; 0 &amp; 1/2 &amp; 0 &amp; 0 \ 1/3 &amp; 0 &amp; 1/3 &amp; 0 &amp; 1/3 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>No</td>
<td>$C = \log_2 3 \approx 1.6605$</td>
<td>$P^* = [1/3, 1/3, 1/3]$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 2/3 &amp; 1/6 &amp; 1/6 \ 2/3 &amp; 1/6 &amp; 1/6 \end{bmatrix}$</td>
<td>No</td>
<td>$C = 0$</td>
<td>$P^* = [1/2, 1/2, 1/2]$</td>
</tr>
</tbody>
</table>

Check that:
1. all the rows of $Q$ are permutations of each other
2. all the column sums are equal

In this problem, $|X|=2$. Therefore, $p$ can be any row vector of length 2 that represents a pmf which means $P_1, P_2 \geq 0$ and $P_1 \cdot P_2 = 0$. 

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